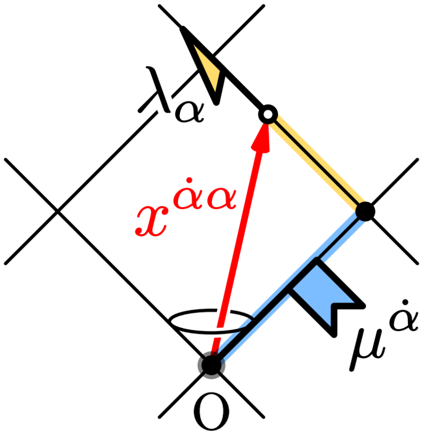
Twistor Theory:

From a “Quantum Particle Theory” Perspective

Joonhwi@Caltech 10/25/2022
4th Floor Journal Club



Arguably, the geometrical and physical significance of null directions in 1+3d general relativity should be appreciated before one starts to think of a theory of quantum gravity. In particular, the fact that the celestial sphere is a Riemann sphere motivates us to dream of a reformulation of 1+3d physics in the “space of light rays” which actively pursues the aesthetics of complex geometry. Twistor theory was born from such a pool of ideas. I systematically derive a dictionary between spacetime and the twistor space by defining the twistor space as the phase space of (a bosonic model of) a massless spinning particle and performing a “phase space matching” via symmetries. Then I discuss helicity amplitudes and their “half Fourier transform” from the first-quantized point of view. Since the conformal group acts linearly on the twistor space, the symmetries of amplitudes become evident when transformed into the twistor space. Twistor string theory and twistor diagrams are briefly discussed. Overall, the theory promotes “radical” rethinking of physics such as regarding spacetime points as secondary constructs or viewing spin angular momentum as an imaginary displacement. However, at the same time, it is a considerably “conservative” approach, appreciating the key features of both quantum and gravitational physics and specializing in four dimensions.

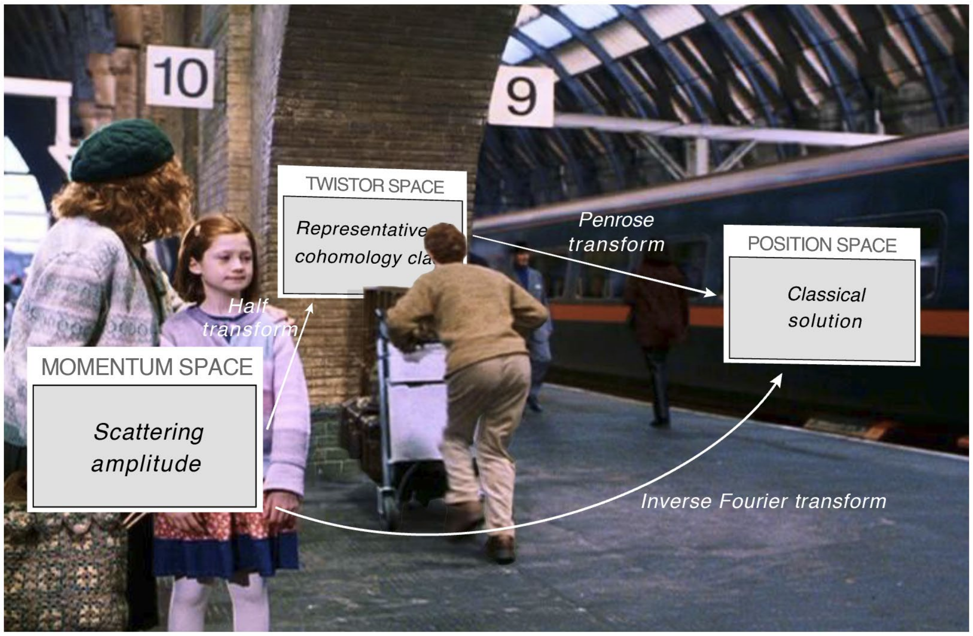


Figure 2: Scheme proposed in ref. [59] for obtaining position-space classical solutions from momentum-space scattering amplitudes in (2,2) signature. The form of the "half transform" is explained in the main text.

Original figure adapted from 2208.08548

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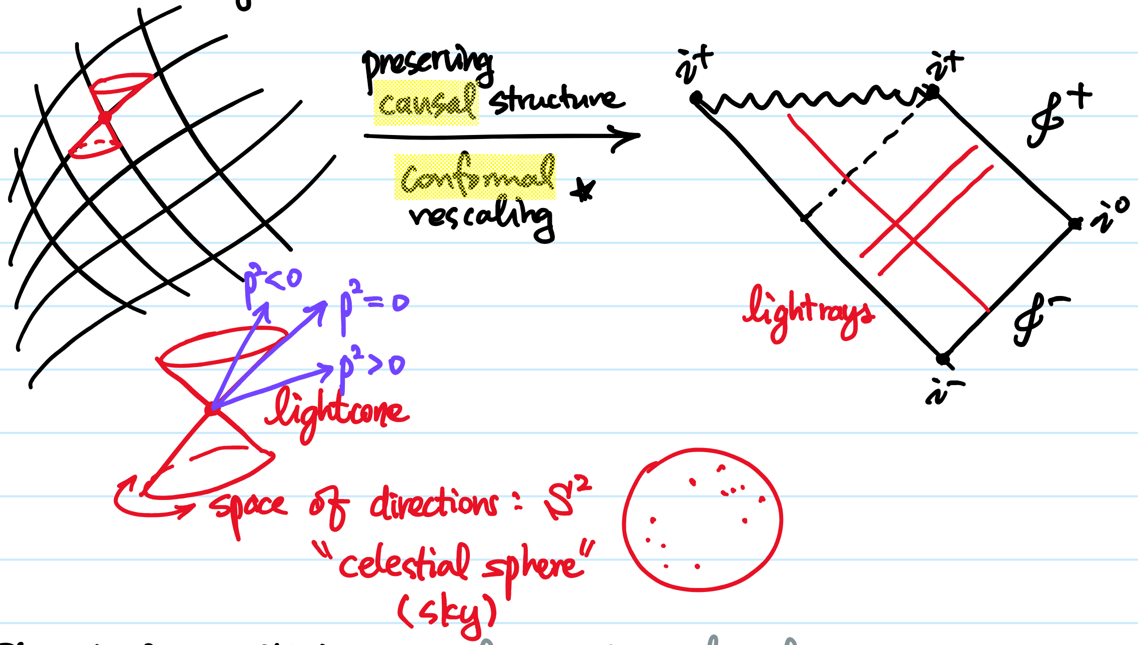
Twistor Theory

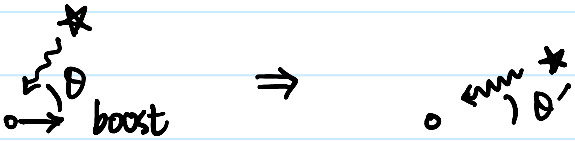
0 Introduction

* The Beauty of 4-dimensional Lorentzian Spacetime

✓ Penrose Diagram

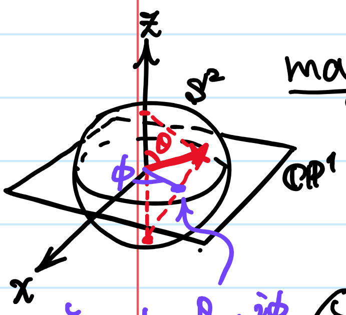


✓ $SO(1,3) \cong C(2)$ cf. embedding formalism



aberration of light

$$\begin{pmatrix}1\\ \cos\theta'\\ \sin\theta'\end{pmatrix} \propto \begin{pmatrix}\gamma & \gamma v & \\ \gamma v & \gamma & \\ & & 1\end{pmatrix}\begin{pmatrix}1\\ \cos\theta\\ \sin\theta\end{pmatrix} = \begin{pmatrix}\gamma(1+v\cos\theta)\\ \gamma(v+\cos\theta)\\ \sin\theta\end{pmatrix}$$



$\zeta = \tan\frac{\theta}{2} e^{i\phi}$

hints

magically $\tan\frac{\theta'}{2} = \frac{\sin\theta'}{1+\cos\theta'} = \frac{\frac{1}{\gamma}\sin\theta}{(1+v)(1+\cos\theta)} = \sqrt{\frac{1-v}{1+v}}\tan\frac{\theta}{2}$

half-angle

beautiful pattern of stars!

rotation

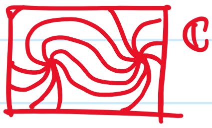
boost

null rotation

cf. hairy ball

generic Lor. transf.

stereographic proj.



$\mathbb{CP}^1$

conformal = holomorphic

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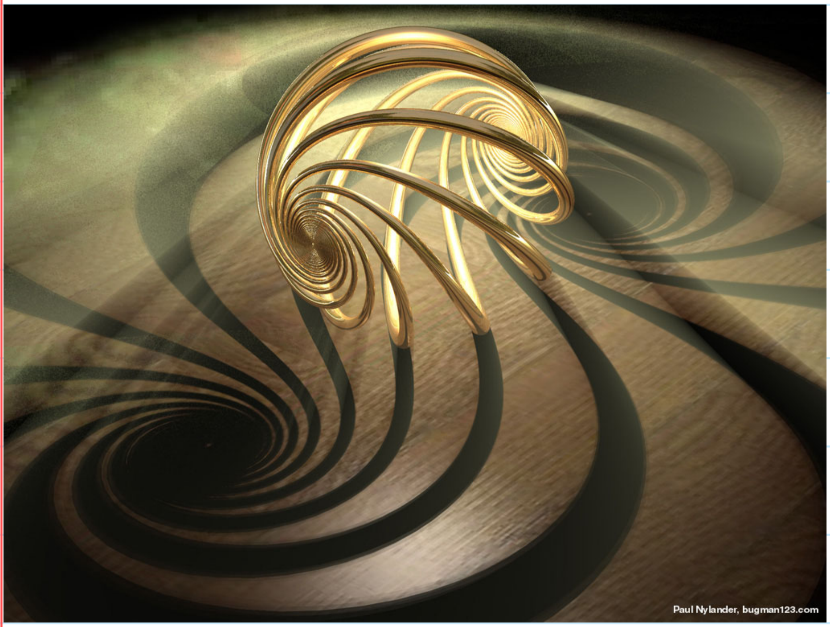
Paul Nylander



Penrose

Spinors and space-time

VOLUME I

TWO-SPINOR CALCULUS AND RELATIVISTIC FIELDS

R.PENROSE & W.RINDLER

CAMBRIDGE MONOGRAPHS ON MATHEMATICAL PHYSICS

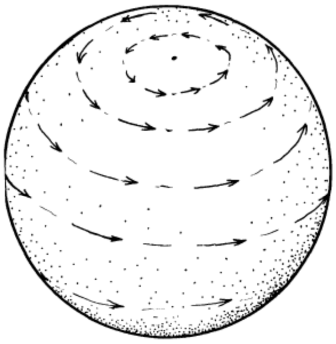


Fig. 1-6. The effect of a rotation on S^+.

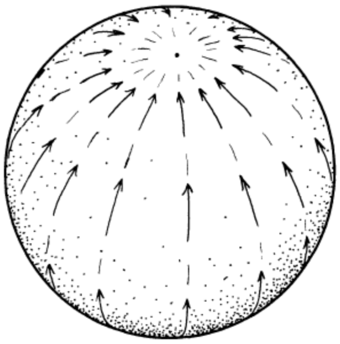


Fig. 1-7. The effect of a boost on S^+.

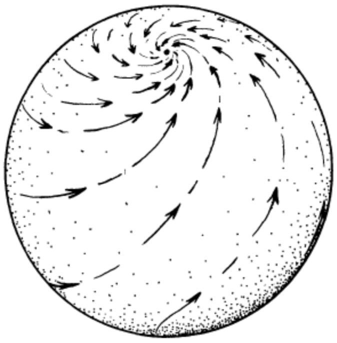
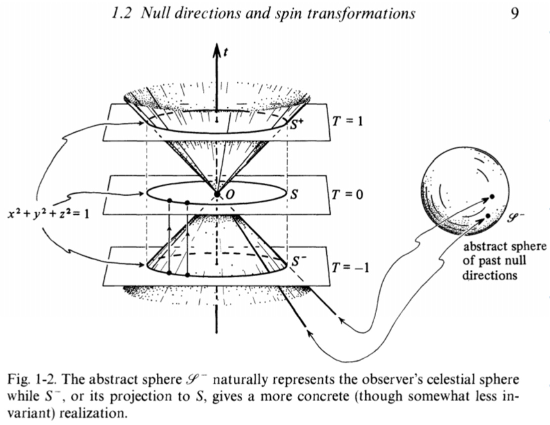


Fig. 1-8. The effect of a four-screw on S^+.

Fig. 1-9. The effect of a null rotation on S^+.



1.2 Null directions and spin transformations 9

Fig. 1-2. The abstract sphere $\mathscr{S}^-$ naturally represents the observer's celestial sphere while S^-, or its projection to S, gives a more concrete (though somewhat less invariant) realization.

My rendering

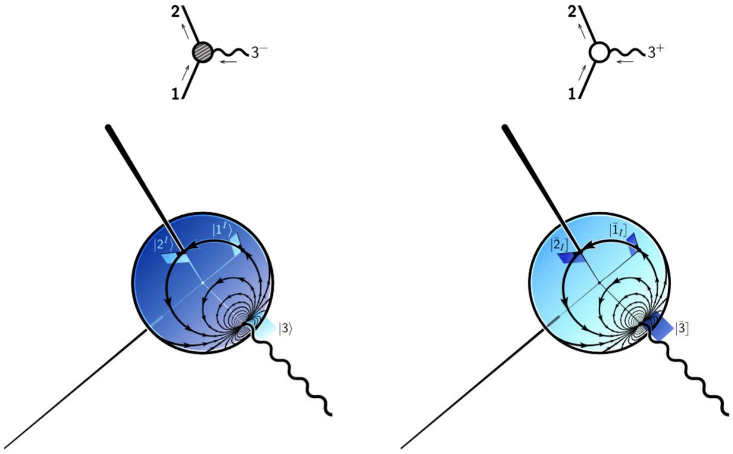


Figure 2. Null rotation on the left and right skies. As a conformal transformation, it preserves the angle that the null flag of the spinor being transformed makes with the circular flow lines. This provides a geometrical way of seeing that $\langle 32^I\rangle = \langle 31^I\rangle$ or $[\bar{2}_I\bar{3}] = [\bar{1}_I\bar{3}]$ [227]. fig:celestial

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* Complex Magic

✓ Laplace eqn in 2d

- $-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi(x,y)=0 \xrightarrow{\text{"}\sqrt{\ }\text{"}} \overset{\text{first-order}}{\boxed{\frac{\partial}{\partial\bar{z}}\psi(x,y)=0}}$

  $z=x+iy \quad \bar{z}=x-iy$

- $\underset{\uparrow\text{harmonic}}{\phi(x,y)} = \text{Re}\,\underset{\uparrow\text{any hol. fn}}{\psi(x+iy)}$

✓ 1+3d wave eqn?

- $\underbrace{\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial z^2}\right)}_{\Box := -\eta^{\mu\nu}\partial_\mu\partial_\nu}\phi(t,x,y,z)$

- Lightcone coordinates? cf. 1+1d wave eqn: lightcone coord $t\pm x$

  very physical: photons travelling

  $f(t+x)+g(t-x)$
  left-movers right-movers

  $$x^{\dot\alpha\alpha} := \frac{1}{-2}(\bar\sigma_\mu)^{\dot\alpha\alpha}x^\mu = \frac{1}{2}\begin{pmatrix} t+z & x-iy \\ x+iy & t-z\end{pmatrix}$$

  Null vectors $(1,0,0,\pm1)$, $(0,1,\pm i,0)$ $\quad$ $+i$ $+$hel, $-i$ $-$hel

  ↑ spatial but null !!! complexification of $\mathbb{E}^{1,3}$

- **Bateman (1904)**:

  $$\phi(t,x,y,z) = \oint\frac{d\zeta}{2\pi i}\tilde{f}(\zeta,(t+z)+(x-iy)\zeta,(x+iy)+(t-z)\zeta)$$

  ↓ projective version

  $$= \oint_{\mathbb{CP}^1}\frac{\langle d\lambda\,\lambda\rangle}{2\pi i}\underset{\uparrow\text{homogeneous in }|\lambda\rangle\text{ of weight }-2}{f^{-2}}(\overset{\lambda_\alpha}{|\lambda\rangle},\overset{x^{\dot\alpha\alpha}\lambda_\alpha}{x|\lambda\rangle})$$

  Moreover...
  $$\psi_\alpha(x) = \oint_{\mathbb{CP}^1}\frac{\langle d\lambda\,\lambda\rangle}{2\pi i}\lambda_\alpha f^{-3}(|\lambda\rangle,x|\lambda\rangle)$$

  ↳ "Penrose-Ward transform" ↙

  $\langle BA\rangle := B^\alpha A_\alpha = \epsilon^{\alpha\beta}A_\alpha B_\beta$

  $[\bar{A}\bar{B}] := \bar{A}_{\dot\alpha}\bar{B}^{\dot\alpha} = \bar\epsilon^{\dot\alpha\dot\beta}\bar{A}_{\dot\alpha}\bar{B}_{\dot\beta}$

- Why? $\partial^{\dot\alpha\alpha}\partial_{\alpha\dot\beta} = \delta^{\dot\alpha}{}_{\dot\beta}\,\Box$

  $$\epsilon^{\alpha\beta}\partial_{\alpha\dot\alpha}\partial_{\beta\dot\beta}\phi(x) = \epsilon^{\alpha\beta}\oint_{\mathbb{CP}^1}\frac{\langle d\lambda\,\lambda\rangle}{2\pi i}\cancel{\lambda_\alpha\lambda_\beta}f_{,\dot\alpha\dot\beta}(|\lambda\rangle,x|\lambda\rangle)$$

  $$=0.$$

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CCFT..

* Theory of Nature that ...

- appreciates the beauty of / significance null directions; geometry of light rays; causal structures; conformal structures in 1+3d

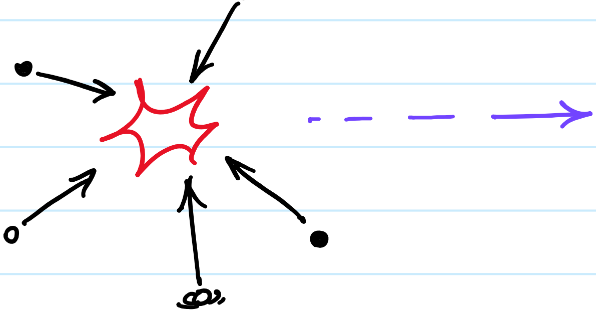
  conformal complex geometry of celestial sphere

  holomorphy

- formulates physics in terms of complex geometry

* Physical Rationale/Motivation?

Quantum physics



"jewel at the heart of quantum physics"

probability amplitude (of scattering)

a complex number $\mathcal{A}$

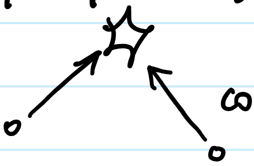
✓ bulk Spacetime is doomed ⟨ AdS/CFT ~ 1974, 1997 / S-matrix program ~ 1960's

qzn of gravity in...

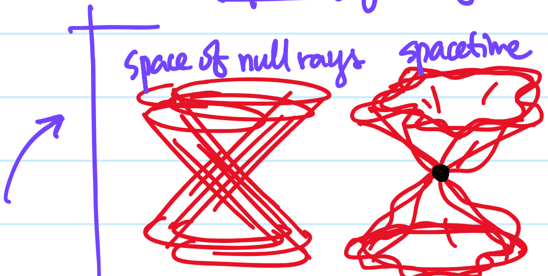
relativists

particle physicists

physically, ...



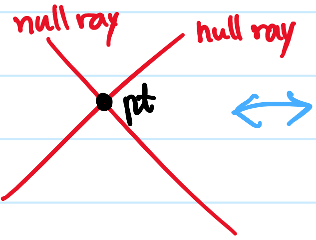
ST points "defined" by measured constructed coincidence of particles



lightrays are fundamental. spacetime points are fundamental

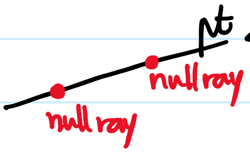
spacetime points "dissolve" disentangle

spacetime



↔

space of null rays



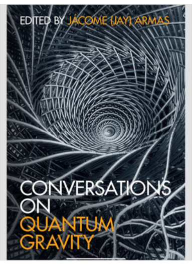
← "secondary construct"

"Space-time is doomed. There is no such thing as space-time fundamentally in the actual underlying description of the laws of physics. That's very startling because what physics is supposed to be about is describing things as they happen in space and time. So if there's no space-time, it's not clear what physics is about." Nima Arkani-Hamed,

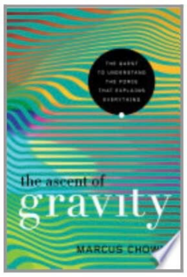
> Nima Arkani-Hamed (06:09): "Almost all of us believe that space-time doesn't really exist, space-time is doomed and has to be replaced by some more primitive building blocks."

Is this related to holography?

It sounds suspiciously like the holographic principle, right (laughs)? In a sense, it's the only really sharp version of the holographic principle (laughs). In fact, it was realised long before Gerard 't Hooft and Leonard Susskind [6, 7]. People who thought deeply about quantum gravity in the 1960s, such as Roger Penrose and Bryce DeWitt, realised this point. Penrose had the idea that there was no bulk spacetime. He didn't quite phrase it so poetically but he really understood that this was the point, namely, that you shouldn't talk about individual points in spacetime because of quantum fluctuations. He suggested that instead you should look at things that go out to infinity such as light waves, which I think was a big motivation for twistor theory [8, 9]. Bryce DeWitt understood that in asymptotically flat spacetime the



Most physicists agree with Wheeler that, on the smallest scales, space-time does not exist. 'Space-time is doomed – that much is pretty universally agreed,' says Nima Arkani-Hamed of the Institute for Advanced Study in Princeton, New Jersey. 'It must be replaced by more fundamental building blocks. The question is what exactly?'



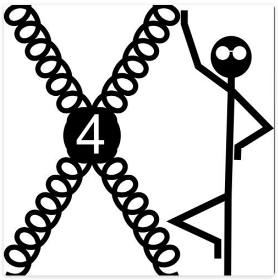
twistor space?

What Can Replace Space-Time?

3 Replies

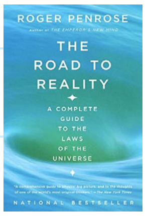
Nima Arkani-Hamed is famous for believing that space-time is doomed, that as physicists we will have to abandon the concepts of space and time if we want to find the ultimate theory of the universe. He's joked that this is what motivates him to get up in the morning. He tends to bring it up often in talks, both for physicists and for the general public.

The latter especially tend to be baffled by this idea. I've heard a lot of questions like "if space-time is doomed, what could replace it?"



Space-time is doomed, and we don't know yet what's going to replace it. But whatever it is, whatever form it takes, we do know one thing: it's going to be a relation between events.

More radical perspectives; twistor theory §33.2



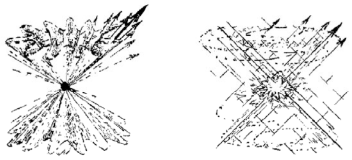

(a) (b)

Fig. 33.7 (a) It has been a common viewpoint, with regard to the possible nature of a 'quantized spacetime', that it should be some kind of a spacetime with a 'fuzzy' metric, leading to some sort of 'fuzzy' light cone, where the notion of a direction at a point being null, timelike, or spacelike would be subject to quantum uncertainties. (b) A more 'twistorial' perspective would be to take the twistor space (in this case $\mathbb{PN}$) to retain some kind of existence (so there would still be light rays), but the condition of their intersection would become subject to quantum uncertainties. Accordingly the notion of 'spacetime point' would instead become 'fuzzy'.

1 Introduction

[1711.09102]

Scattering amplitudes are arguably the most basic observables in fundamental physics. Apart from their prominent role in the experimental exploration of the high energy frontier, scattering amplitudes also have a privileged theoretical status as the only known observable of quantum gravity in asymptotically flat space-time. As such it is natural to ask the "holographic" questions we have become accustomed to asking (and beautifully answering) in AdS spaces for two decades: given that the observables are anchored to the boundaries at infinity, is there also a "theory at infinity" that directly computes the S-Matrix without invoking a local picture of evolution in the interior of the spacetime?

Of course this question is famously harder in flat space than it is in AdS space. The (exceedingly well-known) reason for this is the fundamental difference in the nature of the boundaries of the two spaces. The boundary of AdS is an ordinary flat space with completely standard notions of "time" and "locality", thus we have perfectly natural candidates for what a "theory on the boundary" could be—just a local quantum field theory. We do not have these luxuries in asymptotically flat space. We can certainly think of the "asymptotics" concretely in any of a myriad of ways by specifying the asymptotic on-shell particle momenta in the scattering process. But whether this is done with Mandelstam invariants, or spinor-helicity variables, or twistors, or using the celestial sphere at infinity, in no case is there an obvious notion of "locality" and/or "time" in these spaces, and we are left with the fundamental mystery of what principles a putative "theory of the S-Matrix" should be based on.

Indeed, the absence of a good answer to this question was the fundamental flaw that doomed the 1960's S-Matrix program. Many S-Matrix theorists hoped to find some sort of first-principle "derivation" of fundamental analyticity properties encoding unitarity and causality in the S-Matrix, and in this way to find the principles for a theory of the S-Matrix. But to this day we do not know precisely what these "analyticity properties encoding causality" should be, even in perturbation theory, and so it is not surprising that this "systematic"

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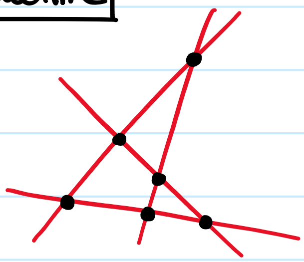
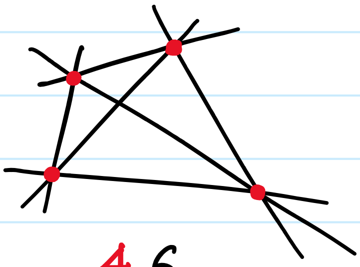
Spacetime



$6_2\, 4_3$

Space of light rays



$4_3\, 6_2$

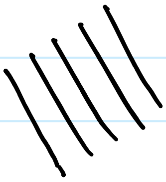
[Configuration (geometry) - Wikipedia](#)
[Spin-Coupling-Diagrams-and-Incidence-Geometry-A-Note-on-Combinatorial-and-Quantum-Computational-Aspects](#)

[1] "Quantum Particle Theory" of Massless Spinning Particles

à la Wigner

* Ptl = irrep. of Poincaré

✓ wave ⟷ ptl



rep. of U(1) ⟶ helicity

$0 \xrightarrow{} p_\mu \overset{*}{=} (-E, \underbrace{0, 0}_{SO(2) \cong U(1)}, E)$

$-{}^*J^{\mu\nu}p_\nu$

little group

have conformal symmetry

Noether charges $(p_\mu, J^{\mu\nu}) \longrightarrow$ on-shell datum (p_μ, s)

spacetime point...

− helicity (ASD)

$$\partial^{\dot\alpha\alpha_1}\phi_{\alpha_1\cdots\alpha_{2s}}(x) = 0 \qquad p^2 = 0, \quad \underset{\text{"ssc"}}{p_\mu S^{\mu\nu} = 0}, \quad {}^*S^{\mu\nu} = -iS^{\mu\nu}$$

$\Downarrow$

$\Box = 0$

$\Uparrow$

defines orbital/spin splitting

$J^{\mu\nu} = \underbrace{x^\mu p^\nu - p^\mu x^\nu}_{L^{\mu\nu}} + S^{\mu\nu}$

+ helicity (SD)

$$\partial^{\dot\alpha_1\alpha}\phi_{\dot\alpha_1\cdots\dot\alpha_{2s}}(x) = 0 \qquad p^2 = 0, \quad p_\mu S^{\mu\nu} = 0, \quad {}^*S^{\mu\nu} = +iS^{\mu\nu}$$

how to reproduce?
(first-qzn)

$$\left[\begin{array}{l} p^2 = 0, \quad p_\mu S^{\mu\nu} = 0 \\ \Rightarrow \; 0 = p_\rho J^{\rho[\mu}p^{\nu]} = \frac{3}{2}p_\rho J^{[\mu\nu}p^{\rho]} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}p_\rho \underbrace{{}^*J_{\sigma\kappa}p^\kappa}_{\propto p_\sigma} \Rightarrow \underline{{}^*J^{\mu\nu}p_\nu = -s\,p^\mu} \end{array}\right]$$

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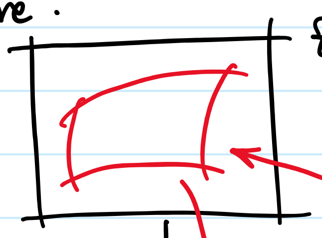
* Helicity Zero

✓ $S[x] = \int d\tau \, -m\sqrt{-\dot{x}^2}$ massive

↓ introduce Lag. mult.

$S[x,p,e] = \int d\sigma \; p_\mu \dot{x}^\mu - e\,\frac{1}{2}(p^2 \cancel{+m^2})$ phase space action

✓ Picture :



8-dim phase space

↓ What $\int \frac{\mathcal{D}e}{\mathrm{vol(Gauge)}}$ does

on-shell

6-dim constrained phase space

↑ symplectic mfds are even-dim

⇒ need "gauge-fixing condition"

"gauge symmetry hits twice" (Teitelboim)

L² of a Lag submfd

$\mathcal{H} \supset \mathcal{H}_{\text{on-shell}}$

constrained Hilbert space

✓ $|x\rangle,\ |p\rangle \in \mathcal{H}$

✓ $|\phi\rangle \in \mathcal{H}_{\text{on-shell}}$ $\Rightarrow \frac{1}{2}\hat{p}^2|\phi\rangle = 0$

$\Rightarrow \langle x|\hat{p}^2|\phi\rangle = 0$

$\Rightarrow \Box \underbrace{\langle x|\phi\rangle}_{\phi(x)} = 0$ KG eqn.

✓ Propagator

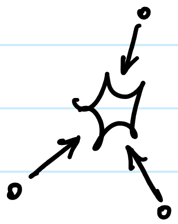
$$\int \frac{\mathcal{D}e}{\mathrm{vol(Gauge)}} \int \mathcal{D}x \int_{p_i}^{p_f} \mathcal{D}p \; e^{-\frac{i}{\hbar}\int d\sigma \, p_\mu \dot{x}^\mu - e\frac{1}{2}p^2}$$

$$= \underbrace{\int_0^\infty dT \; e^{-T(-\frac{i}{\hbar}p^2)}}_{\frac{-i\hbar}{p^2}} \; \delta^{(4)}(p_f - p_i)$$

✓ Amplitudes

"Hardy's compositional principle"

$\mathcal{A}\Big($



$\Big) = [1|\,\hat{S}\,|2]\,|3] = S_{ijk}\, 1^i 2^j 3^k$

on-shell states

"kinematic indizes" ((p,s) in general)

S-matrix is a tensor.

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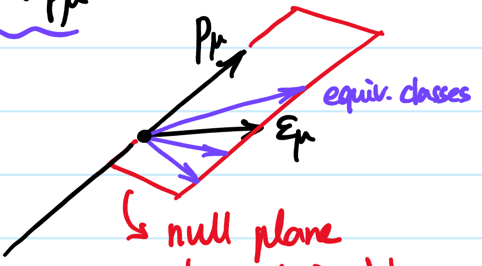
* Helicity One (Photon)

✓ What is a photon, geometrically? (anti-) self-dual a null flag.

(p_μ, ε_μ) with $p\cdot\varepsilon = 0$, $\varepsilon_\mu \sim \varepsilon_\mu + k p_\mu$ → actually a null plane

$(E,0,0,E)$ definite helicity: null

$(k,1,\pm i,k)$

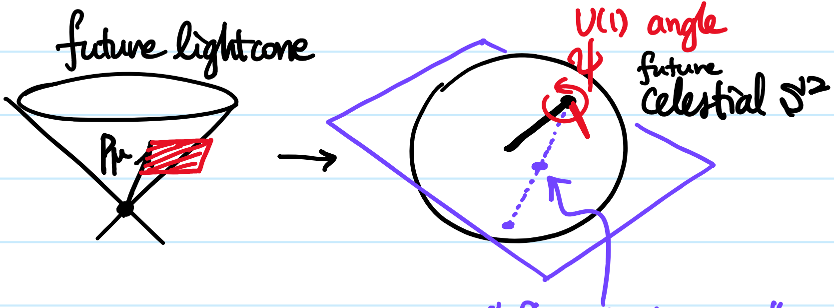


null plane characterised by $P_{[\mu}\varepsilon_{\nu]}$ (field strength!)

✓ Geometric object $\overset{\text{"rep"}}{\longleftrightarrow}$ Algebraic system

"language engineering"

⇝ spinor-helicity variables



$$p_{\alpha\dot\alpha} = -\lambda_\alpha \bar\lambda_{\dot\alpha}$$

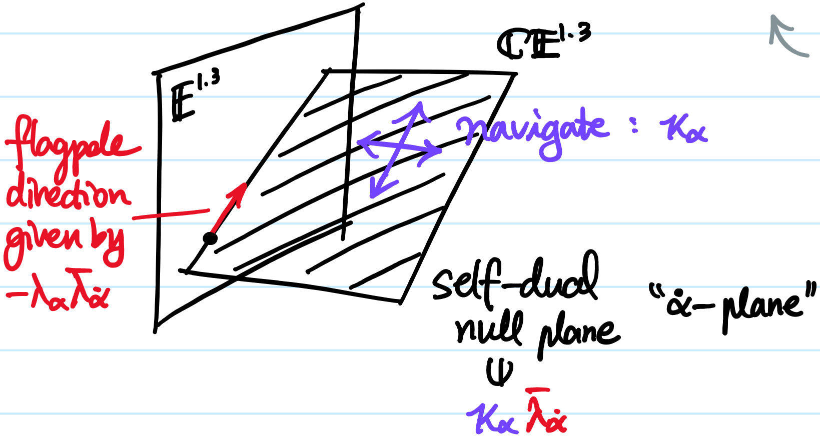
null momentum

"flagpole direction" ↔ $\tan\frac{\theta}{2}e^{i\phi} \in \mathbb{CP}^1$

$$\bar\lambda^{\dot\alpha} \doteq \begin{pmatrix} \cos\frac{\theta}{2}e^{(\psi+\phi)/2i} \\ \sin\frac{\theta}{2}e^{(\psi-\phi)/2i} \end{pmatrix} \longmapsto \tan\frac{\theta}{2}e^{i\phi} \in \mathbb{CP}^1 \cong S^2$$

$$SU(2) \cong S^3 \xrightarrow[\text{fibers over}]{} S^2$$

✓ Null polarization vectors $\varepsilon^+_{\alpha\dot\alpha} = \frac{\overset{\text{reference}}{\eta_\alpha}\bar\lambda_{\dot\alpha}}{\langle\eta\lambda\rangle}$, $\varepsilon^-_{\alpha\dot\alpha} = \frac{\lambda_\alpha \overset{\text{reference}}{\bar\eta_{\dot\alpha}}}{[\bar\eta\bar\lambda]}$



normalization:

$$\epsilon^{\alpha\beta}\frac{1}{-4}\left[2i\left(-\lambda_\alpha\bar\lambda_{\dot\alpha}\right)\frac{\eta_\beta\bar\lambda_{\dot\beta}}{\langle\eta\lambda\rangle}\right]$$

$$= -\frac{1}{2i}\bar\lambda_{\dot\alpha}\bar\lambda_{\dot\beta}$$

SD field strength

⇕

null flag

self-dual null translations $\bar\lambda^{\dot\alpha}\frac{\partial}{\partial x^{\alpha\dot\alpha}}$

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* Classical Worldline Theory of Massless Spinning Particles?

canonical conjugate

✓ $$S[\underline{x,p,\psi,s,e,\kappa}] = \int d\sigma\; p_\mu \dot{x}^\mu + s\dot{\psi} - e\tfrac{1}{2}p^2 - \kappa(s\underbrace{\cancel{-1}}_{\text{for photon, for instance}})$$

10-dim PS ↓ mass-shell 8-dim CPS

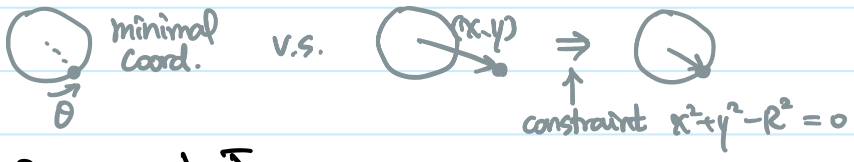
$\begin{cases} 3 \text{ on-shell momentum} \\ 1 \text{ helicity} \end{cases}$

U(1) phase variable for spinning dof

helicity ↓ quantized in the quantum theory.

drop: becomes a "universal spinning pld generator"

✓ Why not explicitly solve the mass-shell constraint?

- 

- $p_{\alpha\dot\alpha} = -\lambda_\alpha \bar\lambda_{\dot\alpha}$

$$\Rightarrow S[\overset{4}{x},\overset{2}{\lambda},\overset{2}{\bar\lambda},\overset{1}{\psi},\overset{1}{s}] = \int d\sigma\; -\bar\lambda_{\dot\alpha}\dot{x}^{\dot\alpha\alpha}\lambda_\alpha + s\dot\psi$$ Shirafuji (1983)

- Problems:

1) Still too much dof $4+2+2+1+1=10$

→ The symplectic structure is <u>degenerate</u>

$$\theta = -\bar\lambda\, dx\, \lambda + s\, d\psi$$

$$\omega = -d\bar\lambda \wedge dx\, \lambda + \bar\lambda\, dx \wedge d\lambda + ds \wedge d\psi$$

ω annihilated by a vector field $-\bar\lambda^{\dot\alpha}\lambda^{\alpha}\frac{\partial}{\partial x^{\dot\alpha\alpha}}$

⇒ quotient the PS ⇒ 8-dim CPS

Minimal coordinates on the CPS? without introducing auxiliary references?

→ $x^{\dot\alpha\alpha}\lambda_\alpha$ and $\bar\lambda_{\dot\alpha}x^{\dot\alpha\alpha}$ are invariant under $x^{\dot\alpha\alpha} \mapsto x^{\dot\alpha\alpha} - k\,\bar\lambda^{\dot\alpha}\lambda^{\alpha}$

2) $-\bar\lambda\dot{x}\lambda$ and $s\dot\psi$ are not "unified" (aesthetic)

not treated on an equal footing

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* Define Hamiltonian System from Symmetry

✓ Conformal symmetry of a massless particle :

$$C(1,3) \cong SO(2,4) \xleftarrow[\text{cover}]{\text{double}} SU(2,2)$$

$$SU(2,2) \xrightarrow{\text{fundrep.}} \underline{\mathbb{C}^4 \text{ equipped w/ } (2,2)\text{-sig Herm. metric}}$$

↑ where gamma matrices act ; Dirac bispinors !

✓ Complex coords. Z_A, $\bar{Z}^A$ of $\mathbb{C}^4$

- Hermitian metric $\bar{Z}^A = [Z_B]^* A^{\bar{B}A}$ ← pseudo-real

$$A^{\bar{A}B} \doteq \begin{pmatrix} 0 & \delta^{\dot\alpha}{}_{\dot\beta} \\ \delta_\alpha{}^\beta & 0 \end{pmatrix} \xrightarrow{\text{diagonalise}} (2,2) \text{ sig.}$$

block

- Chirality structure

$$(\gamma_*)_A{}^B := \frac{1}{4!}\epsilon_{\mu\nu\rho\sigma}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)_A{}^B = \begin{pmatrix} -i\delta_\alpha{}^\beta & 0 \\ 0 & +i\delta^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}$$

→ invariant under origin-fixing conformal transf.s (D, M)

(transformation of a bivector : given by that of total angular momentum ⇒ origin shift can mix SD/ASD components)

- Infinity structure

$$I^{AB} = \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix} , \quad \bar{I}_{AB} = \begin{pmatrix} 0 & 0 \\ 0 & \bar\epsilon_{\dot\alpha\dot\beta} \end{pmatrix}$$

→ invariant under Poincaré (P, M)

⇒ infinitesimal conformal transformation matrix

$$\begin{pmatrix} +\underset{1}{\tfrac{1}{2}\epsilon}\,\overset{\text{AS}}{\delta_\alpha{}^\beta} + \overset{\text{S}}{\underset{3}{\theta_\alpha{}^\beta}} & i\overset{4}{b_{\alpha\dot\beta}} \\ i\underset{4}{a^{\dot\alpha\beta}} & -\tfrac{1}{2}\epsilon\,\delta^{\dot\alpha}{}_{\dot\beta} - \underset{3}{\bar\theta^{\dot\alpha}{}_{\dot\beta}} \end{pmatrix} \leftarrow \text{generic traceless matrix}$$

$1+3+3+4+4 = 16$ ✓

"$(t_a)^i{}_j$"
anti-Herm. generators
traceless

$$\overset{4\times4}{\delta_A{}^D\delta_C{}^B} - \overset{\downarrow\ -1}{\tfrac{1}{4}\delta_A{}^B\delta_C{}^D}$$

$$\underbrace{\alpha_D{}^C}_{(\epsilon,\theta,\bar\theta,a,b)}\overbrace{(t_C{}^D)_A{}^B} = \left(\epsilon\underline{\mathcal{D}} + \theta_{\gamma\delta}\underline{\mathcal{J}^{\gamma\delta}} + \bar\theta_{\dot\gamma\dot\delta}\underline{\mathcal{J}^{\dot\gamma\dot\delta}} + a^{\dot\gamma\gamma}\underline{\mathcal{P}_{\gamma\dot\gamma}} + b_{\gamma\dot\gamma}\underline{\mathcal{K}^{\dot\gamma\gamma}}\right)_A{}^B$$

matrices

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(Schwinger)

✓ Oscillator model

$$\{Z_A, Z_B\} = 0 \qquad \{Z_A, \bar{Z}^B\} = -i\,\delta_A{}^B \qquad \{\bar{Z}^A, \bar{Z}^B\} = 0$$

so that

$$[\hat{Z}_A, \hat{Z}_B] = 0 \qquad [\hat{Z}_A, \hat{\bar{Z}}^B] = \delta_A{}^B \qquad [\hat{\bar{Z}}^A, \hat{\bar{Z}}^B] = 0$$

$$\rightarrow S[\overset{4}{Z}, \overset{4}{\bar{Z}}] = \int d\sigma\ \tfrac{i}{2}(\bar{Z}_A \dot{Z}^A - \dot{\bar{Z}}_A Z^A) ,$$

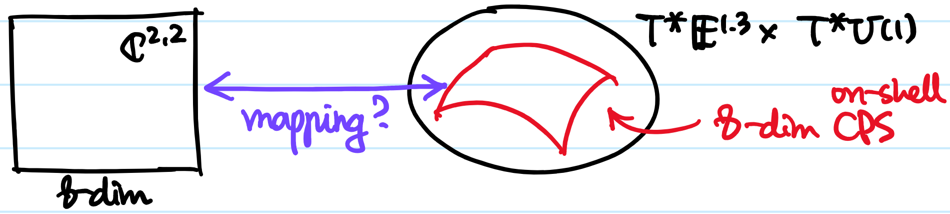
8-dim : Nice!!

$$\theta = \tfrac{i}{2}(\bar{Z}dZ - d\bar{Z}Z)$$

$$\omega = i\, d\bar{Z} \wedge dZ$$

* "Phase Space Matching"

✓



↳ use Noether charges

✓ Conformal symmetry generators

$$Q_C{}^D = i\,\bar{Z}^A (t_C{}^D)_A{}^B Z_B$$

cf. Schwinger

$$\hat{S}_a = \tfrac{\hbar}{2}\, \hat{a}^{\dagger I} (\sigma_a)_I{}^J \hat{a}_J = i\hbar\, \hat{a}^{\dagger I} (t_a)_I{}^J \hat{a}_J$$

$$\Rightarrow \{Z_A, Q_C{}^D\} = (t_C)_A{}^B Z_B , \quad \{\bar{Z}^A, Q_C{}^D\} = -\bar{Z}^B (t_C{}^D)_B{}^A ,$$

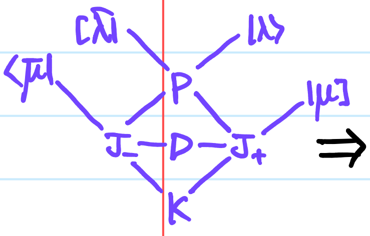
$$\{Q_A{}^B, Q_C{}^D\} = i\,\bar{Z}^E ([t_A{}^B, t_C{}^D])_E{}^F Z_F$$

$$= f_F{}^E{}_A{}^B{}_C{}^D\, Q_E{}^F = \delta_A{}^D Q_B{}^C - Q_A{}^D \delta_B{}^C .$$

$$✓\ i\, \underbrace{(-i\bar{\mu}^\alpha \ \ \bar{\lambda}_{\dot{\alpha}})}_{\bar{Z}^A} \begin{pmatrix} +\tfrac{1}{2}\varepsilon\,\delta_\alpha{}^\beta - \theta_\alpha{}^\beta & i b_{\alpha\dot{\beta}} \\ i a^{\dot{\alpha}\beta} & -\tfrac{1}{2}\varepsilon\,\delta^{\dot{\alpha}}{}_{\dot{\beta}} + \bar{\theta}^{\dot{\alpha}}{}_{\dot{\beta}} \end{pmatrix} \underbrace{\begin{pmatrix} \lambda_\beta \\ i\mu^{\dot{\beta}} \end{pmatrix}}_{Z_B}$$

$$\| $$

$$\varepsilon D + \theta_{\alpha\beta} J^{\alpha\beta} + \theta_{\dot{\alpha}\dot{\beta}} J^{\dot{\alpha}\dot{\beta}} + a^{\dot{\alpha}\alpha} P_{\alpha\dot{\alpha}} + b_{\alpha\dot{\alpha}} K^{\dot{\alpha}\alpha}$$



$$\Rightarrow D = \tfrac{1}{2}(\langle\bar{\mu}\lambda\rangle + [\tilde{\lambda}\mu]) , \quad J^{\alpha\beta} = \bar{\mu}^{(\alpha}\lambda^{\beta)} , \quad J^{\dot{\alpha}\dot{\beta}} = \bar{\lambda}^{(\dot{\alpha}}\mu^{\dot{\beta})} ,$$

$$P_{\alpha\dot{\alpha}} = -\lambda_\alpha \bar{\lambda}_{\dot{\alpha}} , \quad K^{\dot{\alpha}\alpha} = -\mu^{\dot{\alpha}}\bar{\mu}^\alpha$$

✓

inversion : $\lambda \leftrightarrow \mu$

dilatation weight $-\tfrac{1}{2}$ $+\tfrac{1}{2}$

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✓ Spacetime points? $\mathbb{E}^{1,3} = ISO(1,3)/SO(1,3)$ coset

- reference configuration

$$\begin{pmatrix}\lambda_\alpha\\ 0\end{pmatrix} \xrightarrow[\substack{\text{translation}\\ \begin{pmatrix}0 & 0\\ ix^{\dot\alpha\beta} & 0\end{pmatrix}}]{} \begin{pmatrix}\lambda_\alpha\\ ix^{\dot\alpha\beta}\lambda_\beta\end{pmatrix} = \begin{pmatrix}\lambda_\alpha\\ i\mu^{\dot\alpha}\end{pmatrix}$$

$$\Rightarrow \boxed{\mu^{\dot\alpha} = x^{\dot\alpha\alpha}\lambda_\alpha}$$ incidence relation

* My secret trick

Inversion $\not{x} \mapsto \frac{1}{\not{x}}$

SCT $\not{x} \mapsto \frac{1}{\frac{1}{\not{x}}+\not{b}} = \frac{1}{1+\not{x}\not{b}}\not{x}$

$= \not{x} - \underbrace{\not{x}\not{b}\not{x}} + \cdots$

- Flagpole direction of $x^{\dot\alpha\alpha}\lambda_\alpha$

$$\frac{1}{-2}\bar\lambda_{\dot\beta}\, x^{\dot\beta\alpha}(\sigma^\mu)_{\alpha\dot\alpha}\, x^{\dot\alpha\beta}\lambda_\beta$$

$$= \frac{1}{2}(\sigma^\mu)_{\alpha\dot\alpha}(xpx)^{\dot\alpha\alpha}$$

$$= \frac{1}{8}(\sigma^\mu)_{\alpha\dot\alpha}(\sigma^\nu)^{\dot\alpha\alpha}(x^2p_\nu - 2p\cdot x\, x_\nu)$$

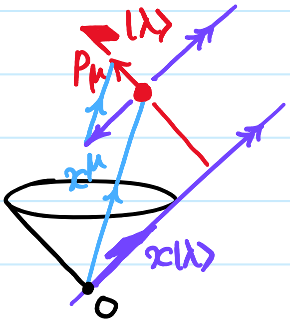
$$= -\frac{1}{4x^2}\left(p^\mu - 2p\cdot x\, x^\mu/x^2\right)$$

"geometric algebra" Doran, …

$$\not{x}\not{p}\not{x} = -2p\cdot x\,\not{x} - \not{x}\not{x}\not{p}$$

$$= -2p\cdot x\,\not{x} + x^2\not{p}$$

inversive/conformal geometry w/ geometric algebra!



geometric meaning of the incidence relation

- $\mu^{\dot\alpha} = x^{\dot\alpha\alpha}\lambda_\alpha$ not invertible

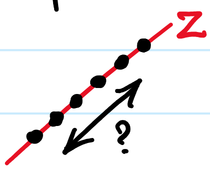
$x^{\dot\alpha\alpha} \sim x^{\dot\alpha\alpha} + \tilde\kappa^{\dot\alpha}\lambda^\alpha$

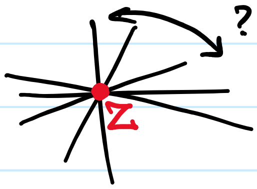
"delocalization" Møller

"the particle dissolves into an α-plane"

$\mathbb{M} = \mathbb{CE}^{1,3}$ spacetime

$\mathbb{T} = \mathbb{C}^{2,2}$ twistor space

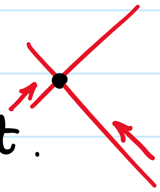


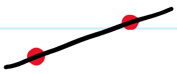


coincidence of two particles

- $\mu^{\dot\alpha I} = x^{\dot\alpha\alpha}\lambda_\alpha{}^I$ ← 1,2 $\Rightarrow x^{\dot\alpha\alpha} = \mu^{\dot\alpha I}(\lambda^{-1})_I{}^\alpha$

a set of two twistors define a spacetime point.





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* Back to the Particle.

✓ $\theta = \frac{i}{2}(\bar{Z}dZ - d\bar{Z}Z)$

$$= \frac{i}{2}(-i\bar{\mu} \quad \bar{\lambda})\begin{pmatrix} d\lambda \\ id\mu \end{pmatrix} - \frac{i}{2}(-id\bar{\mu} \quad d\bar{\lambda})\begin{pmatrix} \lambda \\ i\mu \end{pmatrix}$$

$$= \frac{1}{2}(\bar{\mu}d\lambda - \bar{\lambda}d\mu - d\bar{\mu}\lambda + d\bar{\lambda}\mu)$$

✓ $\omega = d\bar{\mu}^{\alpha} \wedge d\lambda_{\alpha} - d\bar{\lambda}_{\dot{\alpha}} \wedge d\mu^{\dot{\alpha}}$

"momentum"

$$\{\lambda_\alpha, \bar{\mu}^\beta\} = \delta_\alpha{}^\beta \,, \quad \{\bar{\lambda}_{\dot{\alpha}}, \mu^{\dot{\beta}}\} = \delta^{\dot{\beta}}{}_{\dot{\alpha}}$$

"configuration"

$$\mathbb{T} = \mathbb{C}^{2,2} = T^*(\underset{\lambda,\bar{\lambda}}{\mathbb{C}^2})$$

momentum → rather base!

$\{p_\mu, x^\nu\} = -\delta^\nu{}_\mu$

momentum

$\not\neq 0 \to p_\mu$

"x as generator of momentum increment"

bulk position

$$e^{-ik\hat{x}}\,\hat{p}_\mu\, e^{ik\hat{x}} = \hat{p}_\mu + \hbar k_\mu$$

✓ In fact, $\mu^{\dot{\alpha}} = \overset{\mathbb{C}^2}{}z^{\dot{\alpha}\alpha}\overset{\mathbb{C}^2}{\lambda_\alpha}$

can be complex-valued

→ complexified Minkowski space

$\mathbb{CE}^{1,3}$ ($\mathbb{C}^4$ with $\eta_{\mu\nu}$)

becomes (1,3) sig. on the real section

everything complexified: not only $\lambda, \bar{\lambda}$ but also $z, \bar{z}$

✓ Physical meaning of "imaginary displacement"?

"Newman-Janis shift!"

- $J^{\dot{\alpha}\dot{\beta}} = \bar{\lambda}^{(\dot{\alpha}}\mu^{\dot{\beta})} = \bar{\lambda}^{(\dot{\alpha}}z^{\dot{\beta})\beta}\lambda_\beta$

$$= -p_\beta{}^{(\dot{\alpha}}z^{\dot{\beta})\beta}$$

$$\Rightarrow J^{\dot{\alpha}}{}_{\dot{\beta}} = -z^{\dot{\alpha}\alpha}p_{\alpha\dot{\beta}} + \frac{1}{2}\delta^{\dot{\alpha}}{}_{\dot{\beta}}\, p\cdot z$$

$$\bar{J}_\alpha{}^\beta = -p_{\alpha\dot{\alpha}}\bar{z}^{\dot{\alpha}\beta} + \frac{1}{2}\delta_\alpha{}^\beta\, p\cdot\bar{z}$$

$$\Rightarrow J^{\mu\nu} = \text{SD part of } (z^\mu p^\nu - p^\mu z^\nu)$$
$$+ \text{ASD part of } (\bar{z}^\mu p^\nu - p^\mu \bar{z}^\nu)$$
$$= x^\mu p^\nu - p^\mu x^\nu + \underbrace{*(yp-py)^{\mu\nu}}_{\epsilon^{\mu\nu\rho\sigma}y_\rho p_\sigma}$$

$z^\mu := x^\mu + iy^\mu$
$\bar{z}^\mu := x^\mu - iy^\mu$

*
Spin angular momentum "Wick rotates" to Orbital angular momentum in the SD/ASD sectors.

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✓ Helicity.

- $s p^\mu = - * S^{\mu\nu} p_\nu$

$$\Rightarrow s \bar\lambda^{\dot\alpha} \lambda^\alpha = -2\left(-i S^{\dot\alpha\dot\beta} \epsilon^{\alpha\beta} + i \epsilon^{\dot\alpha\dot\beta} \bar S^{\alpha\beta}\right) \frac{1}{-2}(\lambda_\beta \bar\lambda_{\dot\beta})$$

$$= -i \underbrace{S^{\dot\alpha\dot\beta} \bar\lambda_{\dot\beta}}_{\frac{1}{2} i \bar\lambda^{\dot\alpha} [\bar\lambda y \lambda\rangle} \lambda^\alpha + i \bar\lambda^{\dot\alpha} \underbrace{\bar S^{\alpha\beta} \lambda_\beta}_{\frac{1}{2} \cdot -i \lambda^\alpha \langle \bar\lambda y \lambda\rangle}$$

$$= \bar\lambda^{\dot\alpha} \lambda^\alpha [\bar\lambda y \lambda\rangle$$

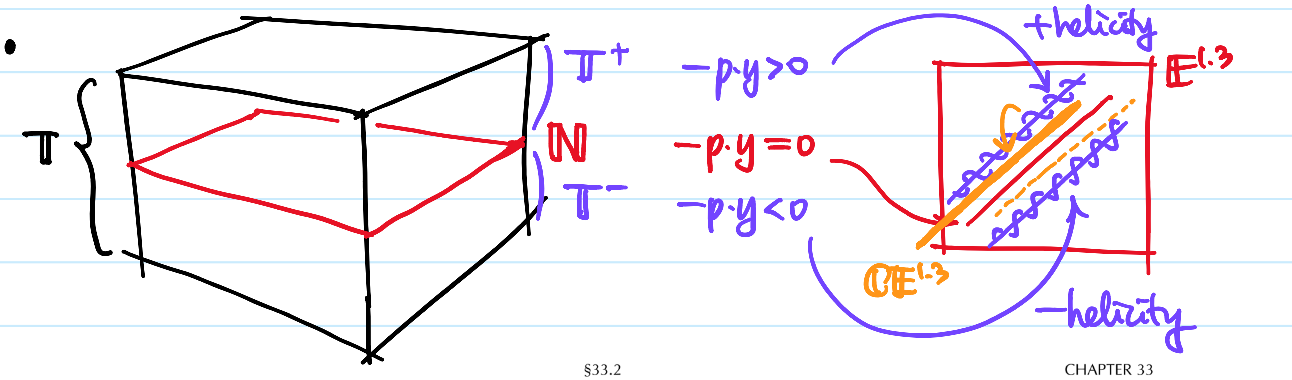
Unification of spin & spacetime

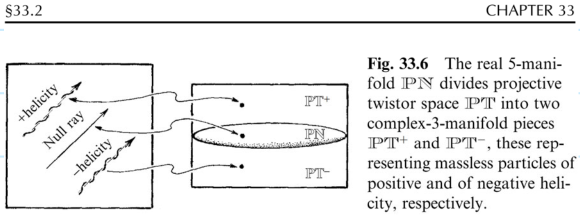
$$\Rightarrow \underset{\sim\sim\sim}{s = -p\cdot y}$$

y^μ : "spin length vector" "spin-space-time"

✓ $\lambda_\alpha \sim e^{-\psi/2 i} \lambda_\alpha$ $\bar\lambda_{\dot\alpha} \sim e^{+\psi/2 i} \bar\lambda^{\dot\alpha}$

$$= \frac{1}{-2} \bar Z^A Z_A \quad : \text{generates } Z_A \mapsto -\frac{1}{2i} Z_A, \ \bar Z^A \mapsto +\frac{1}{2i} \bar Z^A$$

- 



§33.2 CHAPTER 33

Fig. 33.6 The real 5-manifold ℙ𝕅 divides projective twistor space ℙ𝕋 into two complex-3-manifold pieces ℙ𝕋⁺ and ℙ𝕋⁻, these representing massless particles of positive and of negative helicity, respectively.

$$✓ \ \theta = \frac{1}{2}(d\bar\lambda \mu - \bar\lambda d\mu + \bar\mu d\lambda - d\bar\mu \lambda)$$

substitute incidence relations?

$$= \frac{1}{2}\begin{pmatrix} d\bar\lambda z \lambda - \bar\lambda dz \lambda + \bar\lambda \bar z d\lambda - d\bar\lambda \bar z \lambda \\ - \bar\lambda z d\lambda \qquad\qquad - \bar\lambda d\bar z \lambda \end{pmatrix}$$

$$= \underbrace{-\bar\lambda_{\dot\alpha} dx^{\dot\alpha\alpha} \lambda_\alpha}_{p_{\alpha\dot\alpha} dx^{\dot\alpha\alpha}} + \underbrace{i y^{\dot\alpha\alpha} \underline{(\lambda_\alpha d\bar\lambda_{\dot\alpha} - d\lambda_\alpha \bar\lambda_{\dot\alpha})}}_{[\bar\lambda y \lambda\rangle d\psi = s d\psi \ !!}$$

" $\frac{d\psi}{2i} \overset{\circ}{\lambda}_\alpha \overset{\circ}{\bar\lambda}_{\dot\alpha} \times 2$ " ← a reference config

✓

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$\boxed{2}$ The S-matrix in the Twistor Space

* 1st qtn of the twistor particle ⇒ $|\lambda\bar\lambda\rangle$ $|Z\rangle$ etc.

$\langle\lambda\bar\mu|$ $\langle\bar Z|$

✓ Conformal transformations / non-linearly realized in $\langle\lambda\bar\mu|$ polarization (or in spacetime)

linearly realized in $\langle\bar Z|$ polarization (in the twistor space)

$$\begin{cases}\langle\bar Z|\hat Z_A = \frac{\partial}{\partial\bar Z^A}\langle\bar Z| \\ \langle\bar Z|\hat{\bar Z}^A = \bar Z^A\langle\bar Z|\end{cases}$$

- $\hat Q_A{}^B = i\left(\hat{\bar Z}^B\hat Z_A - \frac{1}{4}\delta_A{}^B\hat{\bar Z}_C\hat Z^C\right)$ no ordering issue

$$\langle\bar Z|\hat Q_A{}^B = i\left(\bar Z^B\frac{\partial}{\partial\bar Z^A} - \frac{1}{4}\delta_A{}^B\bar Z_C\frac{\partial}{\partial\bar Z_C}\right)\langle\bar Z|$$

- $\hat D = \frac{1}{2}\left(\hat\mu^\alpha\hat\lambda_\alpha + \hat{\bar\lambda}_{\dot\alpha}\hat\mu^{\dot\alpha}\right) \rightsquigarrow \frac{1}{2i}\left(-\bar\mu^\alpha\frac{\partial}{\partial\bar\mu^\alpha} + \bar\lambda_{\dot\alpha}\frac{\partial}{\partial\bar\lambda_{\dot\alpha}}\right)$

$\hat J^{\alpha\beta} = \hat\mu^{(\alpha}\hat\lambda^{\beta)} \rightsquigarrow -i\bar\mu^{(\alpha}\frac{\partial}{\partial\bar\mu_{\beta)}}$

$\hat{\tilde J}^{\dot\alpha\dot\beta} = \hat{\bar\lambda}^{(\dot\alpha}\hat\mu^{\dot\beta)} \rightsquigarrow -i\bar\lambda^{(\dot\alpha}\frac{\partial}{\partial\bar\lambda_{\dot\beta)}}$

$\hat P_{\alpha\dot\alpha} = -\hat{\bar\lambda}_{\dot\alpha}\hat\lambda_\alpha \rightsquigarrow -i\bar\lambda_{\dot\alpha}\frac{\partial}{\partial\bar\mu^\alpha}$

$-i\frac{\partial}{\partial\bar\lambda}$

$\bar\lambda_\alpha \rightsquigarrow i\frac{\partial}{\partial\bar\mu^a}$

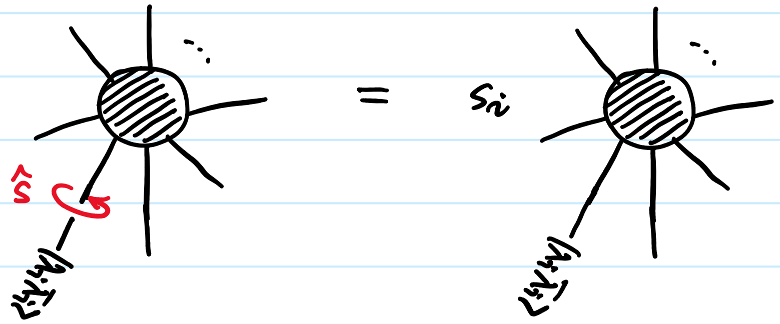
$\hat K^{\dot\alpha\alpha} = -\hat\mu^\alpha\hat\mu^{\dot\alpha} \rightsquigarrow i\bar\mu^\alpha\frac{\partial}{\partial\bar\lambda_{\dot\alpha}}$ $-i\bar\mu^\alpha\bar\lambda_\alpha$

✓ Helicity operator

- $\hat s = \frac{1}{2}\left(\hat{\bar Z}^A\hat Z_A \overset{\text{ordering choice}}{+1}\right) \rightsquigarrow \frac{1}{2}\left(\bar\lambda_{\dot\alpha}\frac{\partial}{\partial\bar\lambda_{\dot\alpha}} + \underbrace{\bar\mu^\alpha\frac{\partial}{\partial\bar\mu^\alpha} + 1}_{-\lambda_\alpha\frac{\partial}{\partial\lambda_\alpha}\text{ in }\langle\lambda\bar\mu|\text{ pol.}}\right)$

* Helicity amplitude

"covariance of a tensor"
"eigenvalue eqn."



for each leg i

... (*)

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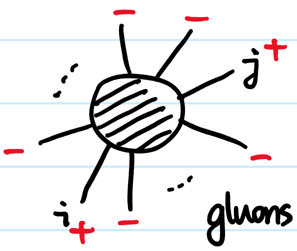
✓ MHV tree : (*) uniquely determines the amplitude → conformally invariant, but hard to see



$$= g^{n-2} \frac{[\bar{i}\bar{j}]^4}{[\bar{1}\bar{2}][\bar{2}\bar{3}]\cdots[\bar{n}\bar{1}]} \delta^{(4)}\left(|1\rangle[\bar{1}| + \cdots |n\rangle[\bar{n}|\right)$$

$\bar{\lambda}$ ↗

? ⇒ secretly go to (2,2) signature

- $$\mathcal{A}(\bar{Z}_1, \cdots \bar{Z}_n) = \underbrace{\left(\prod_{i=1}^{n} \int d^2\lambda_i \, e^{-i\langle\bar{\mu}_i \lambda_i\rangle}\right)}_{\text{“half Fourier transform”}} \mathcal{A}(\lambda_1, \bar{\lambda}_1, \cdots \lambda_n, \bar{\lambda}_n)$$

amplitude in the twistor space

helicity amplitude

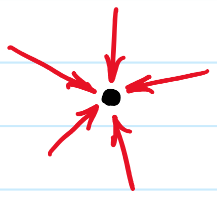
$$= g^{n-2} \frac{[\bar{i}\bar{j}]^4}{[\bar{1}\bar{2}][\bar{2}\bar{3}]\cdots[\bar{n}\bar{1}]} \left(\prod_{i=1}^{n} \int d^2\lambda_i \, e^{-i\langle\bar{\mu}_i \lambda_i\rangle}\right)$$

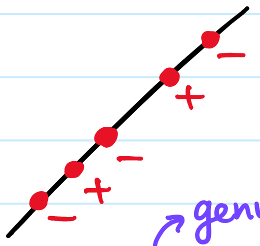
MHV!

$$\int d^4x \; e^{i[\bar{\lambda}_1|x|\lambda_1\rangle + \cdots + i[\bar{\lambda}_n|x|\lambda_n\rangle}$$

$$= g^{n-2} \frac{[\bar{i}\bar{j}]^4}{[\bar{1}\bar{2}][\bar{2}\bar{3}]\cdots[\bar{n}\bar{1}]} \int d^4x \prod_{i=1}^{n} \delta^{(2)}\left(\langle\bar{\mu}_i| - [\bar{\lambda}_i|x\right)$$

localizes on the support of incidence relations sharing the same pt.





↗ genus 0 : $\mathbb{CP}^1 \cong S^2$

$\int d^4x$: integral over the moduli space of - - - ↶ degree 1 holomorphic curve in $\mathbb{CP}^3$ ← $Z/\bar{Z}$ rescaling redundancy

✓ <u>conjecture</u> degree = (# neg helicity) − 1 + (# loops)

⇒ genus ≤ (# loops)

⤳ tempting to interpret the curve as the string worldsheet

“twistor string theory” Witten (2004) ~

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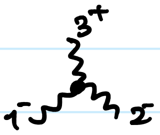
* A closer look : 3pt MHV



$$g\,\frac{[\tilde\lambda_1\tilde\lambda_2]^3}{[\tilde\lambda_2\tilde\lambda_3][\tilde\lambda_3\tilde\lambda_1]}$$

✓ What if...

$$\mathcal{A}(\bar Z_1,\bar Z_2,Z_3)=\underline{\int d^2\lambda_1\,e^{-i\langle\bar\mu_1\lambda_1\rangle}}\ \underline{\int d^2\lambda_2\,e^{-i\langle\bar\mu_2\lambda_2\rangle}}$$

$$\int d^2\tilde\lambda_3\,e^{-i[\tilde\lambda_3\mu_3]}\ \boxed{g\,\frac{[\tilde\lambda_1\tilde\lambda_2]^3}{[\tilde\lambda_2\tilde\lambda_3][\tilde\lambda_3\tilde\lambda_1]}}$$

$$\int d^4x\ \underline{e^{i[\tilde\lambda_1|x|\lambda_1\rangle}\,e^{i[\tilde\lambda_2|x|\lambda_2\rangle}}\ e^{i[\tilde\lambda_3|x|\lambda_3\rangle}$$

$$=g[\tilde\lambda_1\tilde\lambda_2]^3\int d^4x\,\delta^{(2)}(\langle\bar\mu_1|-[\tilde\lambda_1|x)\,\delta^{(2)}(\langle\bar\mu_2|-[\tilde\lambda_2|x)$$

$$\underbrace{\int d^2\tilde\lambda_3\,\frac{1}{[\tilde\lambda_2\tilde\lambda_3][\tilde\lambda_3\tilde\lambda_1]}\,e^{-i[\tilde\lambda_3|(|\mu_3]-x|\lambda_3\rangle)}}$$

$$|\tilde\lambda_3]=a\,|\tilde\lambda_1]+b\,|\tilde\lambda_2]\ \downarrow$$

$$\int|[\tilde\lambda_1\tilde\lambda_2]|\,da\,db\,\frac{1}{[\tilde\lambda_2\tilde\lambda_1]a\,b[\tilde\lambda_2\tilde\lambda_1]}\,e^{-i(a[\tilde\lambda_1|+b[\tilde\lambda_2|)|\mu_3-x\lambda_3]}$$

$$=\frac{\mathrm{sgn}([\tilde\lambda_1\tilde\lambda_2])}{[\tilde\lambda_1\tilde\lambda_2]}\int da\,db\,\frac{1}{ab}\,e^{-ia[\tilde\lambda_1|\cancel{\mu_3-x\lambda_3]}}\,e^{-ib[\tilde\lambda_2|\cancel{\mu_3-x\lambda_3]}}$$

$$[\tilde\lambda_1\mu_3]-\langle\bar\mu_1\lambda_3\rangle=-i\,\bar Z_1Z_3 \qquad [\tilde\lambda_2\mu_3]-\langle\bar\mu_2\lambda_3\rangle=-i\,\bar Z_2Z_3$$

$$=g\,[\tilde\lambda_1\tilde\lambda_2]^2\,\mathrm{sgn}([\tilde\lambda_1\tilde\lambda_2])$$

$$x=\left(|\tilde\lambda_2]\langle\bar\mu_1|-|\tilde\lambda_1]\langle\bar\mu_2|\right)\frac{1}{[\tilde\lambda_1\tilde\lambda_2]}-x'$$

$$\underbrace{\int d^4x\,\delta^{(2)}(\langle\bar\mu_1|-[\tilde\lambda_1|x)\,\delta^{(2)}(\langle\bar\mu_2|-[\tilde\lambda_2|x)}\ \Rightarrow\int d^4x'\,\delta^{(2)}([\tilde\lambda_1|x')\,\delta^{(2)}([\tilde\lambda_2|x')=[\tilde\lambda_1\tilde\lambda_2]^{-2}$$

$$\int\frac{da}{a}\,\frac{db}{b}\,e^{-a\bar Z_1Z_3}\,e^{-b\bar Z_2Z_3}$$

conformal symmetry manifest

$$=g\,\mathrm{sgn}(\underbrace{[\tilde\lambda_1\tilde\lambda_2]}_{\bar Z_1^A\,\bar I_{AB}\,\bar Z_2^B})\,\mathrm{sgn}(\cancel{i}\,\underline{\bar Z_1Z_3})\,\mathrm{sgn}(\cancel{i}\,\bar Z_2Z_3)$$

a.c to (2,2) sig

asymp. states are at infinity? ⇐

"mild" breaking of C.I. : sign changes at colin. sing. (!?)

amplitudes are "1 or −1"

NAH, Cachazo, CC, Kaplan (2009)

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✓ "double copy"

$$\mathbb{A}^{-1-1+1}(\bar{Z}_1,\bar{Z}_2,Z_3) = g\,\mathrm{sgn}(\bar{Z}_1 I \bar{Z}_2)\,\mathrm{sgn}(\bar{Z}_1 Z_3)\,\mathrm{sgn}(\bar{Z}_2 Z_3)$$

$\downarrow$ $\qquad\qquad$ $\downarrow$ "sgn to abs"

$$\mathbb{A}^{-2+2+2}(\bar{Z}_1,\bar{Z}_2,Z_3) = \frac{1}{M_{pl}}\,|\bar{Z}_1 I \bar{Z}_2|\,|\bar{Z}_1 Z_3|\,|\bar{Z}_2 Z_3|$$

C.I. breaking becomes clear

gravity amplitude : infinity twistor plays a central role.

Mason & Skinner (2009) curved twistor theory : Hamiltonian deformation of cpx structure

Penrose & MacCallum (1973) contour integral

more 'classic twistor theorist' approach

✓ "link representation"

"Hodges diagram" A. Hodges (2005)

ACCK(2009) : concrete rel. b/w MHV diagrams

- Z ○ 'zig' $\qquad$ $\bar{Z}$ ● 'zag'

- $\mathrm{sgn}(\bar{Z}_1 Z_2)$ 

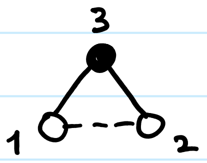
$\bar{Z}_1 Z_2$ 

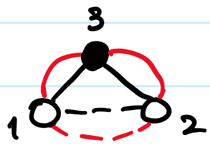
- $\mathrm{sgn}(\bar{Z}_1 I \bar{Z}_2)$ 

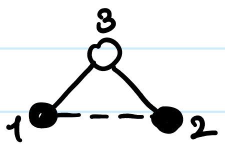
$\bar{Z}_1 I \bar{Z}_2$ 

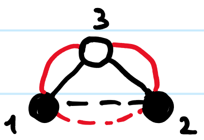
- $e^{i\bar{Z}_1 Z_2}$ 

Fourier transform $\bar{Z} \leftrightarrow Z$

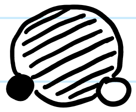
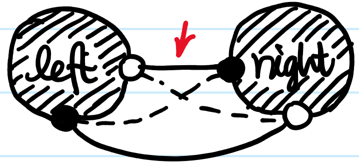
- $\mathbb{A}^{-1-1+1} =$ 

$\mathbb{A}^{-2-2+2} =$ 

- $\mathbb{A}^{+1+1-1} =$ 

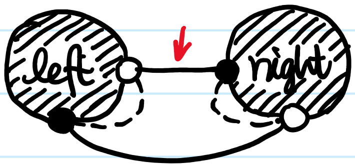
$\mathbb{A}^{+2+2-2} =$ 

* BCFW Recursion in the Twistor Space

✓  = 

gravity: add two — lines here


BCFW bridge

= 

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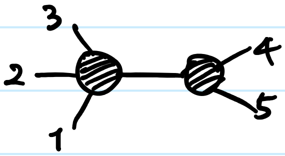
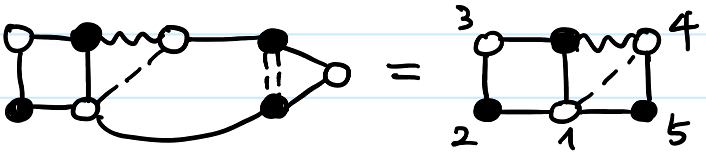
✓ 5pt MHV





R. Penrose and M.A.H. MacCallum, Twistor theory: an approach to the quantisation of fields and space-time 281

Penrose & MacCallum (1973)

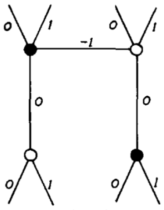


Fig. 14.

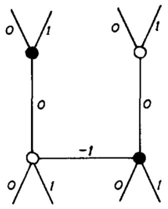


Fig. 15.

allowed to be "off the mass shell", i.e. to have a non-zero (and sometimes imaginary) rest-mass. Particles with non-zero rest-mass do not have a well-defined helicity. In the twistor approach, on the other hand, it would be unreasonable to allow photons to have a rest-mass, since this would go against the basic philosophy of the theory. Nevertheless the theory could not give sensible answers (for Möller scattering, for example) if it did not in some way reflect the fact which, in the conventional formalism, is accounted for by allowing virtual photons to be off the mass shell. This, in itself, renders it unlikely that the twistor computation of Möller scattering could be obtainable as a sum of two integrals, like those represented in figs. 14 and 15, in each of which the contribution due to a virtual photon appears to be identifiable, the photon having a well-defined helicity[25].

↳ BCFW!

already points to the modern on-shell approach to amplitudes! Yutin, Nima, ...

~40-50 years ago...

also...

[1706.02314]

Holomorphic Classical Limit for Spin Effects in Gravitational and Electromagnetic Scattering

Alfredo Guevara

Perimeter Institute for Theoretical Physics, Waterloo, ON N2L 2Y5, Canada

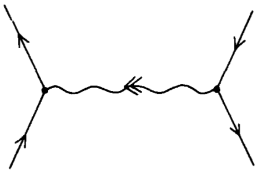


Fig. 16a.

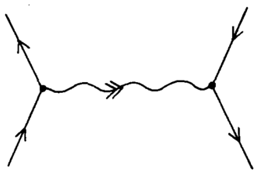


Fig. 16b.

R. Penrose and M.A.H. MacCallum, Twistor theory: an approach to the quantisation of fields and space-time 283

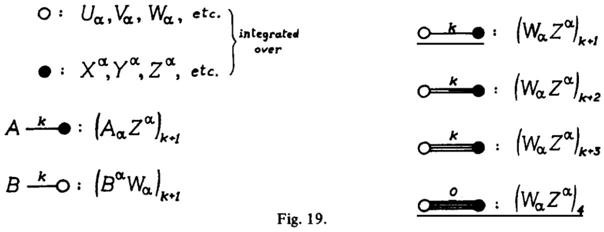


Fig. 19.

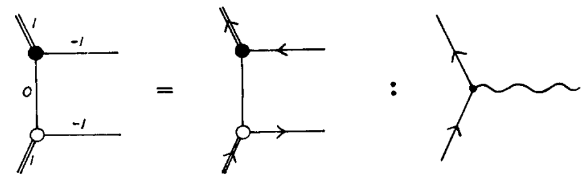


Fig. 28.

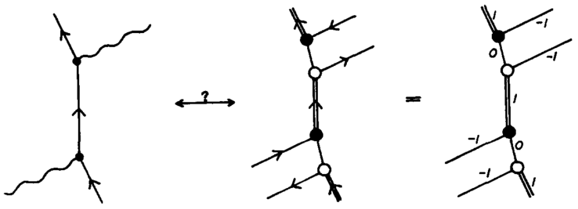


Fig. 29.

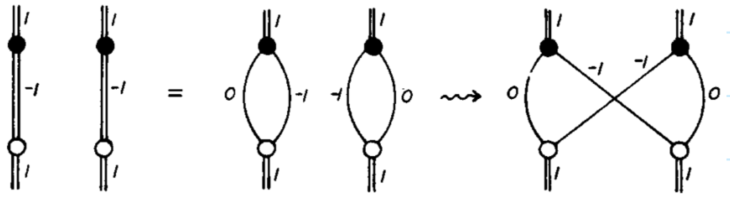


Fig. 25.

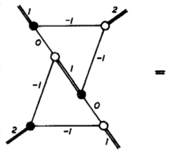


Fig. 30.

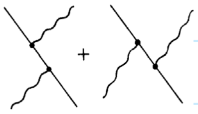
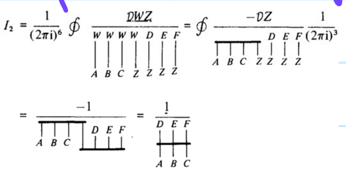


Fig. 31.

defined in terms of twistor contour integral



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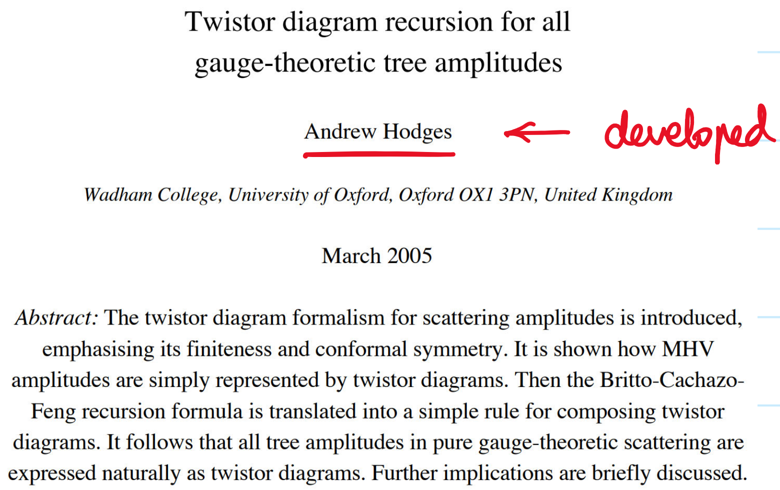


Twistor diagram recursion for all gauge-theoretic tree amplitudes

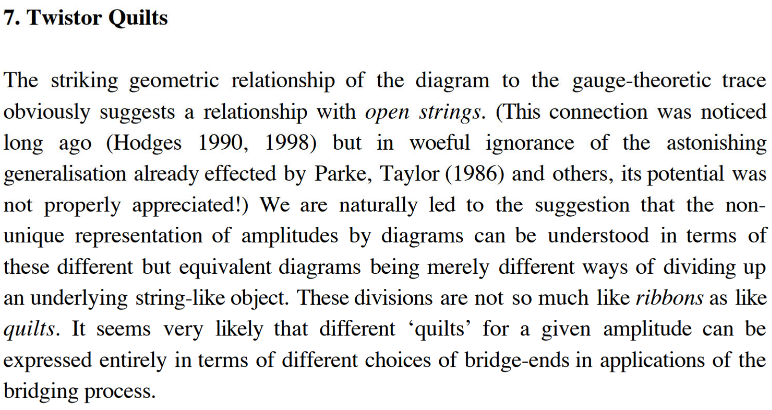
Andrew Hodges

Wadham College, University of Oxford, Oxford OX1 3PN, United Kingdom

March 2005

Abstract: The twistor diagram formalism for scattering amplitudes is introduced, emphasising its finiteness and conformal symmetry. It is shown how MHV amplitudes are simply represented by twistor diagrams. Then the Britto-Cachazo-Feng recursion formula is translated into a simple rule for composing twistor diagrams. It follows that all tree amplitudes in pure gauge-theoretic scattering are expressed naturally as twistor diagrams. Further implications are briefly discussed.

← developed the idea "single-handedly"
1980s ~ 2000s

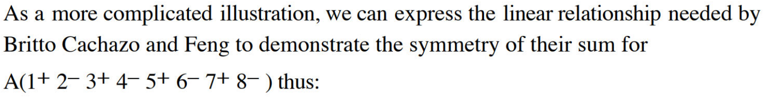


7. Twistor Quilts

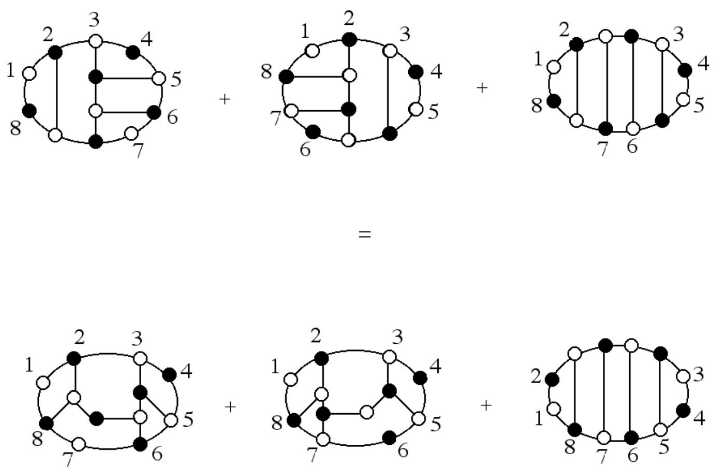
The striking geometric relationship of the diagram to the gauge-theoretic trace obviously suggests a relationship with *open strings*. (This connection was noticed long ago (Hodges 1990, 1998) but in woeful ignorance of the astonishing generalisation already effected by Parke, Taylor (1986) and others, its potential was not properly appreciated!) We are naturally led to the suggestion that the non-unique representation of amplitudes by diagrams can be understood in terms of these different but equivalent diagrams being merely different ways of dividing up an underlying string-like object. These divisions are not so much like *ribbons* as like *quilts*. It seems very likely that different 'quilts' for a given amplitude can be expressed entirely in terms of different choices of bridge-ends in applications of the bridging process.

Photo by Colin Watson, *August 2006.*

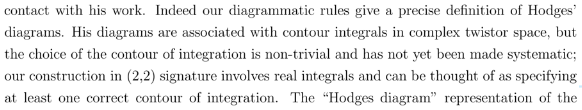
https://www.twistordiagrams.org.uk/



As a more complicated illustration, we can express the linear relationship needed by Britto Cachazo and Feng to demonstrate the symmetry of their sum for $A(1^+ 2^- 3^+ 4^- 5^+ 6^- 7^+ 8^-)$ thus:

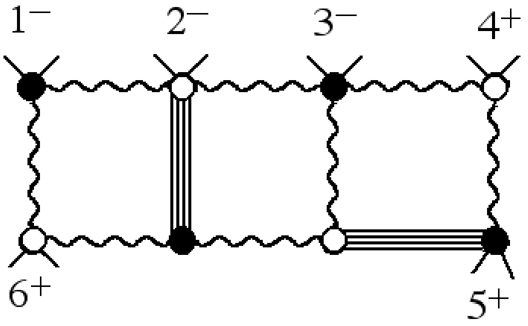


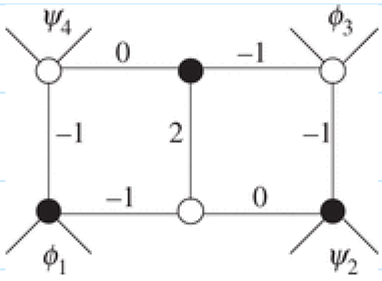
but...
integral not well-defined!



contact with his work. Indeed our diagrammatic rules give a precise definition of Hodges' diagrams. His diagrams are associated with contour integrals in complex twistor space, but the choice of the contour of integration is non-trivial and has not yet been made systematic; our construction in (2,2) signature involves real integrals and can be thought of as specifying at least one correct contour of integration. The "Hodges diagram" representation of the

complex contour integral being only formal





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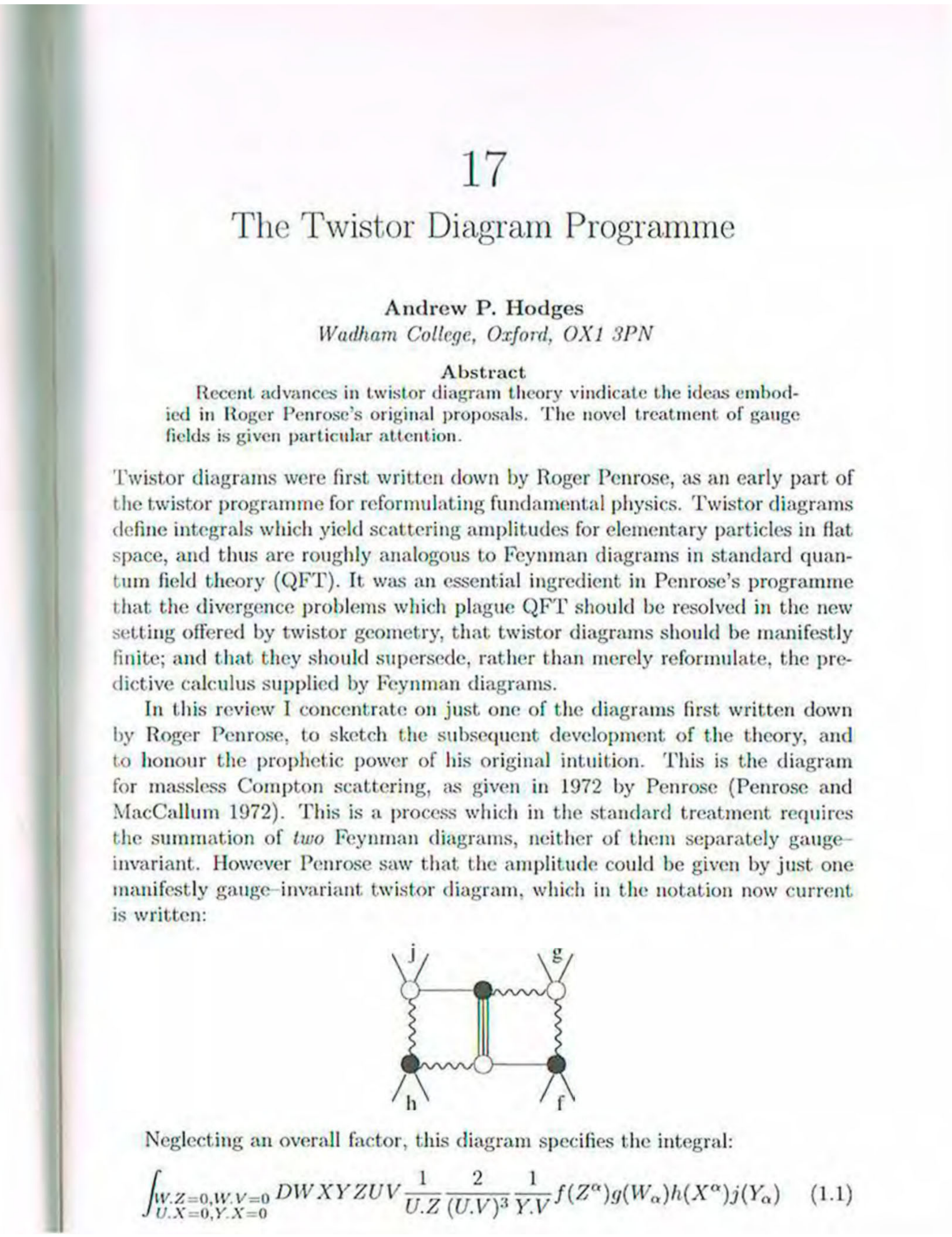


17

The Twistor Diagram Programme

Andrew P. Hodges
Wadham College, Oxford, OX1 3PN

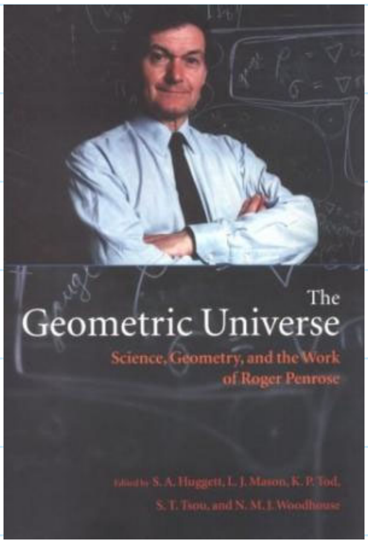
Abstract
Recent advances in twistor diagram theory vindicate the ideas embodied in Roger Penrose's original proposals. The novel treatment of gauge fields is given particular attention.

Twistor diagrams were first written down by Roger Penrose, as an early part of the twistor programme for reformulating fundamental physics. Twistor diagrams define integrals which yield scattering amplitudes for elementary particles in flat space, and thus are roughly analogous to Feynman diagrams in standard quantum field theory (QFT). It was an essential ingredient in Penrose's programme that the divergence problems which plague QFT should be resolved in the new setting offered by twistor geometry, that twistor diagrams should be manifestly finite; and that they should supersede, rather than merely reformulate, the predictive calculus supplied by Feynman diagrams.

In this review I concentrate on just one of the diagrams first written down by Roger Penrose, to sketch the subsequent development of the theory, and to honour the prophetic power of his original intuition. This is the diagram for massless Compton scattering, as given in 1972 by Penrose (Penrose and MacCallum 1972). This is a process which in the standard treatment requires the summation of *two* Feynman diagrams, neither of them separately gauge-invariant. However Penrose saw that the amplitude could be given by just one manifestly gauge-invariant twistor diagram, which in the notation now current is written:

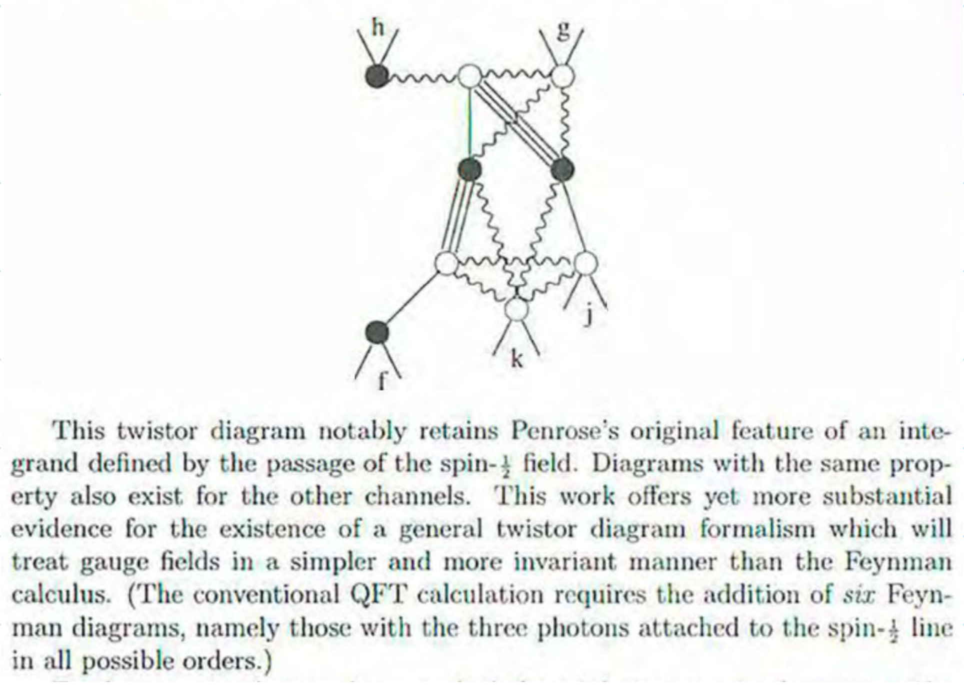
Neglecting an overall factor, this diagram specifies the integral:

$$\int_{\substack{W.Z=0,W.V=0\\ U.X=0,Y.X=0}} DWXYZUV\frac{1}{U.Z}\frac{2}{(U.V)^3}\frac{1}{Y.V}f(Z^\alpha)g(W_\alpha)h(X^\alpha)j(Y_\alpha) \qquad (1.1)$$



The
Geometric Universe
Science, Geometry, and the Work of Roger Penrose
Edited by S. A. Huggett, L. J. Mason, K. P. Tod, S. T. Tsou, and N. M. J. Woodhouse

A. Hodges, Twistor Diagrams, Physica 114A (1982) 157, Twistor Diagrams, in The Geometric Universe: Science, Geometry, and The Work Of Roger Penrose, eds. S. A. Huggett et. al. (Oxford University Press, 1998)



This twistor diagram notably retains Penrose's original feature of an integrand defined by the passage of the spin-$\frac{1}{2}$ field. Diagrams with the same property also exist for the other channels. This work offers yet more substantial evidence for the existence of a general twistor diagram formalism which will treat gauge fields in a simpler and more invariant manner than the Feynman calculus. (The conventional QFT calculation requires the addition of *six* Feynman diagrams, namely those with the three photons attached to the spin-$\frac{1}{2}$ line in all possible orders.)

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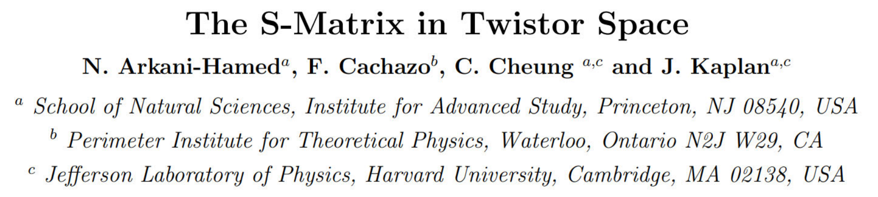
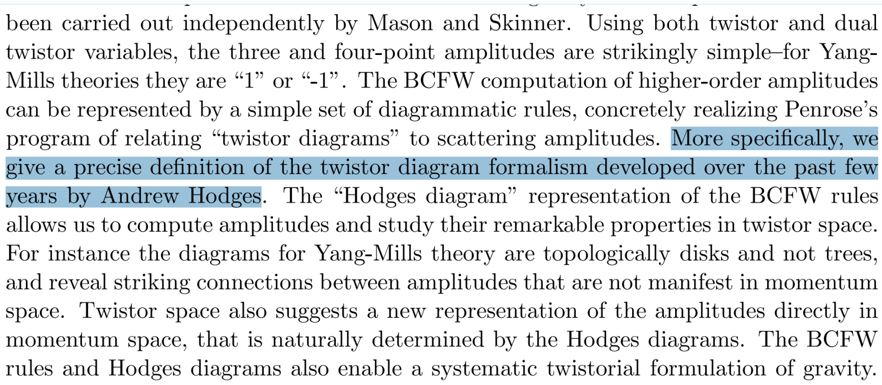


The S-Matrix in Twistor Space

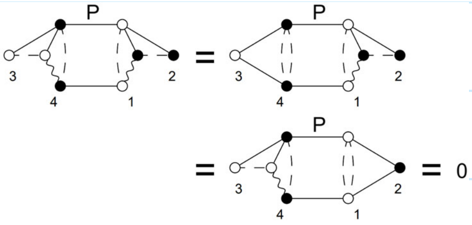
N. Arkani-Hamed[a], **F. Cachazo**[b], **C. Cheung** [a,c] **and J. Kaplan**[a,c]

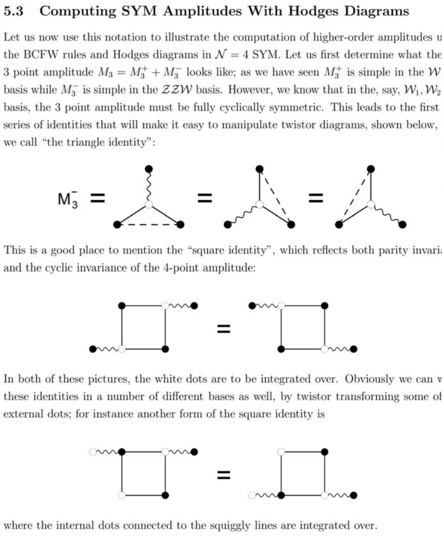
[a] *School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540, USA*
[b] *Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2J W29, CA*
[c] *Jefferson Laboratory of Physics, Harvard University, Cambridge, MA 02138, USA*



been carried out independently by Mason and Skinner. Using both twistor and dual twistor variables, the three and four-point amplitudes are strikingly simple–for Yang-Mills theories they are "1" or "-1". The BCFW computation of higher-order amplitudes can be represented by a simple set of diagrammatic rules, concretely realizing Penrose's program of relating "twistor diagrams" to scattering amplitudes. More specifically, we give a precise definition of the twistor diagram formalism developed over the past few years by Andrew Hodges. The "Hodges diagram" representation of the BCFW rules allows us to compute amplitudes and study their remarkable properties in twistor space. For instance the diagrams for Yang-Mills theory are topologically disks and not trees, and reveal striking connections between amplitudes that are not manifest in momentum space. Twistor space also suggests a new representation of the amplitudes directly in momentum space, that is naturally determined by the Hodges diagrams. The BCFW rules and Hodges diagrams also enable a systematic twistorial formulation of gravity.

concrete definition of Hodges diagrams by going to (2,2) sig





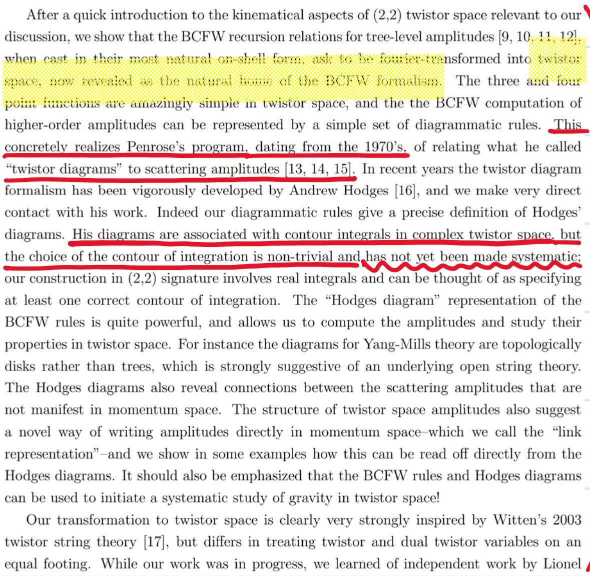
5.3 Computing SYM Amplitudes With Hodges Diagrams

Let us now use this notation to illustrate the computation of higher-order amplitudes u the BCFW rules and Hodges diagrams in $\mathcal{N}=4$ SYM. Let us first determine what the 3 point amplitude $M_3 = M_3^+ + M_3^-$ looks like; as we have seen M_3^+ is simple in the $\mathcal{W}$ basis while M_3^- is simple in the $\mathcal{ZZW}$ basis. However, we know that in the, say, $\mathcal{W}_1, \mathcal{W}_2$ basis, the 3 point amplitude must be fully cyclically symmetric. This leads to the first series of identities that will make it easy to manipulate twistor diagrams, shown below, we call "the triangle identity":

This is a good place to mention the "square identity", which reflects both parity invari and the cyclic invariance of the 4-point amplitude:

In both of these pictures, the white dots are to be integrated over. Obviously we can v these identities in a number of different bases as well, by twistor transforming some of external dots; for instance another form of the square identity is

where the internal dots connected to the squiggly lines are integrated over.

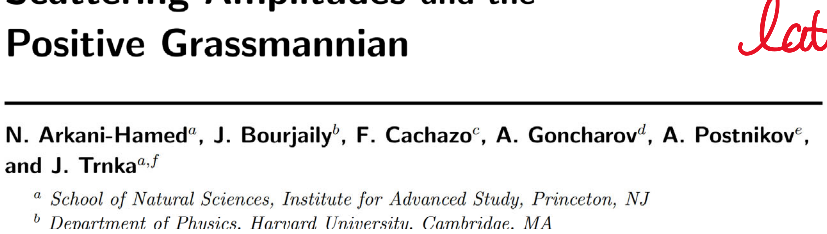


After a quick introduction to the kinematical aspects of (2,2) twistor space relevant to our discussion, we show that the BCFW recursion relations for tree-level amplitudes [9, 10, 11, 12], when cast in their most natural on-shell form, ask to be fourier-transformed into twistor space, now revealed as the natural home of the BCFW formalism. The three and four point functions are amazingly simple in twistor space, and the the BCFW computation of higher-order amplitudes can be represented by a simple set of diagrammatic rules. This concretely realizes Penrose's program, dating from the 1970's, of relating what he called "twistor diagrams" to scattering amplitudes [13, 14, 15]. In recent years the twistor diagram formalism has been vigorously developed by Andrew Hodges [16], and we make very direct contact with his work. Indeed our diagrammatic rules give a precise definition of Hodges' diagrams. His diagrams are associated with contour integrals in complex twistor space, but the choice of the contour of integration is non-trivial and has not yet been made systematic; our construction in (2,2) signature involves real integrals and can be thought of as specifying at least one correct contour of integration. The "Hodges diagram" representation of the BCFW rules is quite powerful, and allows us to compute the amplitudes and study their properties in twistor space. For instance the diagrams for Yang-Mills theory are topologically disks rather than trees, which is strongly suggestive of an underlying open string theory. The Hodges diagrams also reveal connections between the scattering amplitudes that are not manifest in momentum space. The structure of twistor space amplitudes also suggest a novel way of writing amplitudes directly in momentum space–which we call the "link representation"–and we show in some examples how this can be read off directly from the Hodges diagrams. It should also be emphasized that the BCFW rules and Hodges diagrams can be used to initiate a systematic study of gravity in twistor space!

Our transformation to twistor space is clearly very strongly inspired by Witten's 2003 twistor string theory [17], but differs in treating twistor and dual twistor variables on an equal footing. While our work was in progress, we learned of independent work by Lionel

good overview of the history

[1212.5605]

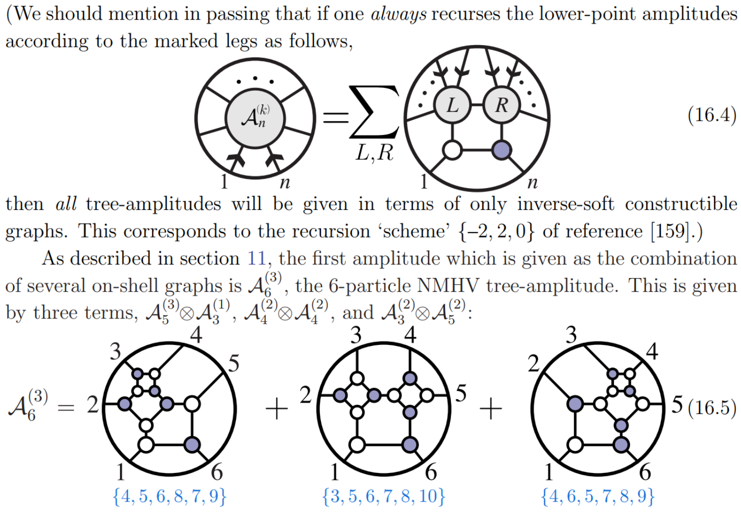


Scattering Amplitudes and the Positive Grassmannian

N. Arkani-Hamed[a], **J. Bourjaily**[b], **F. Cachazo**[c], **A. Goncharov**[d], **A. Postnikov**[e], **and J. Trnka**[a,f]

[a] *School of Natural Sciences, Institute for Advanced Study, Princeton, NJ*
[b] *Department of Physics, Harvard University, Cambridge, MA*

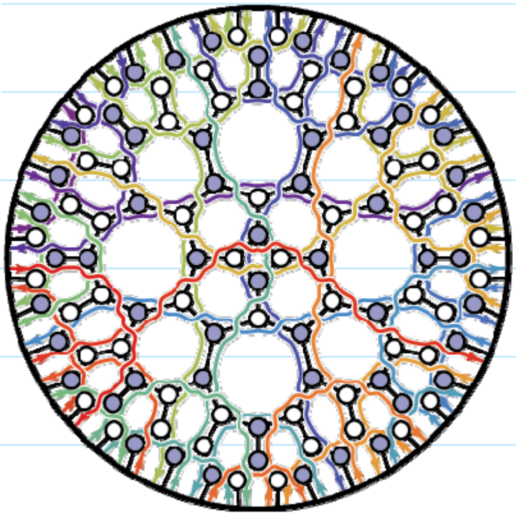
later...



(We should mention in passing that if one *always* recurses the lower-point amplitudes according to the marked legs as follows,

then *all* tree-amplitudes will be given in terms of only inverse-soft constructible graphs. This corresponds to the recursion 'scheme' $\{-2, 2, 0\}$ of reference [159].)

As described in section 11, the first amplitude which is given as the combination of several on-shell graphs is $\mathcal{A}_6^{(3)}$, the 6-particle NMHV tree-amplitude. This is given by three terms, $\mathcal{A}_5^{(3)}\otimes\mathcal{A}_3^{(1)}$, $\mathcal{A}_4^{(2)}\otimes\mathcal{A}_4^{(2)}$, and $\mathcal{A}_3^{(2)}\otimes\mathcal{A}_5^{(2)}$:



Tuesday, October 25, 2022 8:53 AM

"$9\frac{3}{4}$"

*

$$\begin{tikzcd} \boxed{\mathbb{M}} \arrow[rr, bend left, "\text{Fourier}"] & \boxed{\mathbb{P}(\mathbb{T})} \arrow[l, "\text{Penrose-Ward transform}"'] & \boxed{\mathbb{CP}^1\times\mathbb{CP}^1} \arrow[l, "\text{half Fourier}"'] \\ x & Z & \lambda, \tilde{\lambda} \\ & & \wr \end{tikzcd}$$

spacetime twistor space momentum space

$$\phi(Z) = \oint_{\mathbb{CP}^1} \frac{\langle d\lambda\, \lambda\rangle}{2\pi i} \overbrace{f(|\lambda\rangle, Z|\lambda\rangle)}^{\text{Čech cohomology representative}}$$

$$\phi_{\alpha\beta}(Z) = \oint_{\mathbb{CP}^1} \frac{\langle d\lambda\, \lambda\rangle}{2\pi i} f(|\lambda\rangle, Z|\lambda\rangle) \quad \lambda_\alpha \lambda_\beta$$

"Weyl double copy"

$$\phi_{\alpha\beta\gamma\delta}(Z) = \oint_{\mathbb{CP}^1} \frac{\langle d\lambda\, \lambda\rangle}{2\pi i} f(|\lambda\rangle, Z|\lambda\rangle) \quad \lambda_\alpha \lambda_\beta \lambda_\gamma \lambda_\delta$$

↖ twistor space amplitude Guevara (2022)

Non-pert sol.s : BHs, Instantons

\ Newman-Janis

\ Kerr thm $\mathbb{T} \overset{\text{ST}}{\underset{\text{SST}}{\times}}$

a "different" thry of QG ?

Mason Ashtekar

curved twistor theory?

Mason & Skinner: graviton MHV scattering

Earth" problem

Formulate field theories on $\mathbb{T}$?

" twistor actions "

Λ?

↪ I_{AB}

* What is a twistor?

- a light ray : an $\dot{\alpha}$-plane (SD null flag)
- fundrep. of the conformal group $SU(2,2)$
- T^*(spinor-helicity) (T^*(celestial sphere))
- incidence relation? → a soln of $\partial^{\dot{\alpha}}_{\ \beta}\, \omega^{\dot{\alpha}}(x) = 0$

Googly problem

Physics from the space of light rays

a lot more to explore!